Tropical Geometry and Mirror Symmetry

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Introduction to Tropical Geometry

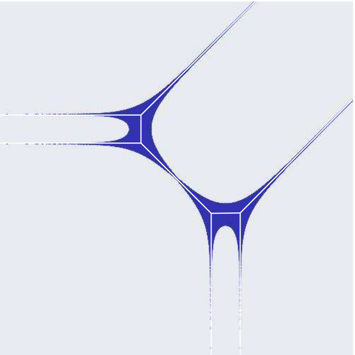
Set $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and consider a conic in $(\mathbb{C}^*)^2$, e.g.

$$C = \{(x, y) \mid f(x, y) = 0\}, \qquad f = 10 + x + y + 10xy + 10^{-1}x^2 + 10^{-1}y^2.$$

The image under

$$\text{Log} : (x, y) \mapsto (\log |x|, \log |y|)$$

Gives the *amoeba* $\text{Log}(C)$



whose spine is obtained from the Ronkin function $\mathbb{R}^2 \to \mathbb{R}$ defined as

$$\frac{1}{(2\pi i)^2} \int_{\text{Log}^{-1}(x,y)} \log |f(x, y)| \frac{dx}{x} \wedge \frac{dy}{y}.$$

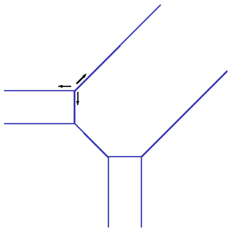
This is convex and even affine linear away from $Log(C)$.

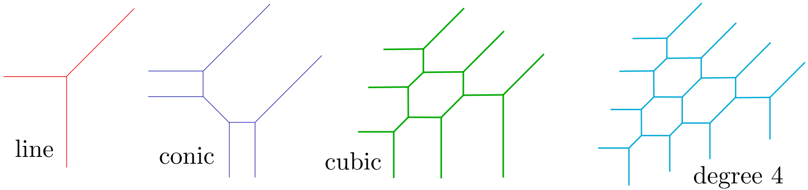
(Ronkin-Passare-Rullgård, Mikhalkin)

Tropical curves

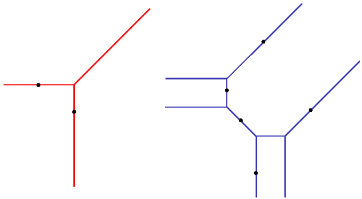
Definition

A *tropical curve* in $\mathbb{R}^2$ is an immersion of a integer-weighted graph with edges mapping to line segments of rational slope such that for every vertex a balancing condition holds, i.e. the weighted sum of the primitive generators of the adjacent edges gives zero.





Tropical Rational Curve Counts



Two points determine a tropical line, five points determine a tropical conic.

Theorem (Mikhalkin,Siebert-Nishinou 2004)

The counts of complex rational degree d curves through $3d-1$ points coincide with the counts of tropical rational curves through $3d-1$.

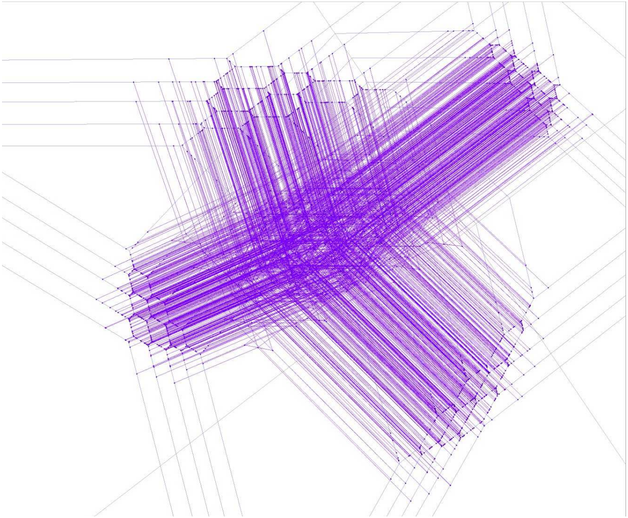
(This holds more generally in smooth toric surfaces for any genus (Mikhalkin) and in higher dimensions for rational curves (Siebert-Nishinou) and for both these situations also with ψ-class conditions (R.-Mandel).

Goal:

Make this work for curves in Calabi-Yau manifolds using toric degenerations!

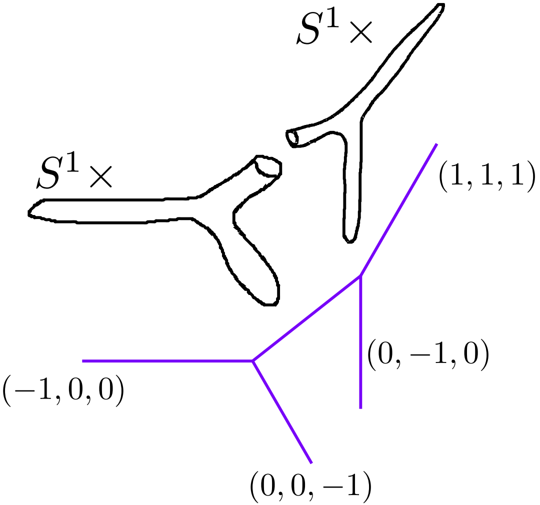
2875 lines on a quintic threefold

A general quintic threefold X in $\mathbb{P}^4$ contains 2875 lines. This proof is due to S.Katz: Degenerate the quintic to the union of the coordinate hyperplanes and find $\frac{2875}{5} = 575$ lines in each hyperplane $\mathbb{P}^3$ as the set of lines that meet the four quintic curves $X \cap \mathbb{P}^2$ but don't meet any coordinate-$\mathbb{P}^1$s.



Lagrangian threefolds in the mirror quintic

The tropical representation allows to construct mirror dual objects to the lines: 2875 pairwise disjoint Lagrangian spheres (j.w. Cheuk Yu Mak). Each sphere fibres over the tropical lines with generic fibre $(S^1)^2$:



Lagrangian Lens Spaces in non-generic quintics

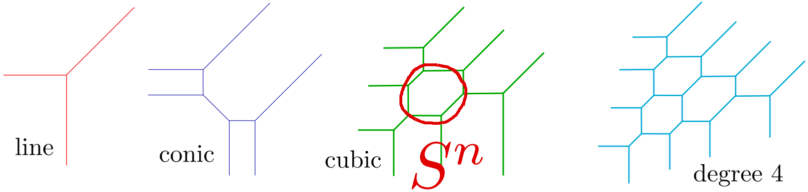
The tropical count is weighted by Mikhalkin-multiplicities: $\sum_\Gamma \mathrm{Mult}(\Gamma)$

For non-generic quintics, we find tropical lines Γ with $p = \mathrm{Mult}(\Gamma) > 1$.

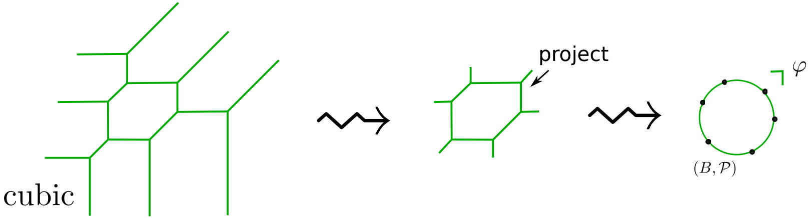
These give rise to Lagrangian lens spaces $L(p, q)$ in the mirror quintic.

- No examples of Lagrangian lens spaces in compact symplectic manifolds were known before.
- One obtains via Seidel-Dehn twists automorphisms of the Fukaya category that interact in interesting ways with taking characteristic p.

What makes Calabi-Yau manifolds special from a tropical point of view



We obtain an integral affine manifold B with polyhedral decomposition $\mathscr{P}$ and multi-valued piecewise linear function φ. Gross-Siebert call the triple $(B, \mathscr{P}, \varphi)$ a *tropical manifold* (not to be confused with Mikhalkin's notion that refers to a different object).



Mirror symmetry from tropical geometry

Discrete Legendre transform provides a duality

$$(B,\mathcal{P},\varphi) \leftrightarrow (\check{B},\check{\mathcal{P}},\check{\varphi})$$

that realizes mirror symmetry as follows.

Theorem (Gross-Siebert)

Given a positive and simple tropical manifold $(B,\mathcal{P},\varphi)$, there is a canonical formal family with general fibre an irreducible Calabi-Yau variety[1]

$$\begin{array}{c}\mathfrak{X}\\ \downarrow\\ \operatorname{Spf} k[H][\![t]\!]\end{array}$$

where $H = H^1(B,\iota_*\check{\Lambda})^*$ for $\iota : B_{\mathrm{reg}} \hookrightarrow B$ the inclusion and Λ the local system of integral tangent vectors on B_{reg}.

[1] field k is algebraically closed of $\operatorname{char} k = 0$

Analyticity of the canonical family

Theorem (R-Siebert)

For $k = \mathbb{C}$, B closed, the canonical family $\mathfrak{X} \to \operatorname{Spf} k[H][\![t]\!]$ is the formalization of an analytic family

$$\begin{array}{c} \mathfrak{X} \\ \downarrow \\ U \end{array}$$

for $U \subset \operatorname{Spec}(k[H][t])_{\mathrm{an}}$, in particular, the monomials in $k[H][t]$ are holomorphic.

Proof: We have a $\mathbb{C}^\times$-action on the family that rescales t. The period integrals $\int_\beta \Omega$ are a complete set of invariants of the action. Periods are known to be holomorphic. Explicit calculation identifies periods with monomials in the base, now the result follows from the existence of versal deformation spaces by Grauert-Douady.

Tropical 1-cycles give period integrals

Theorem (R)

For $(B, \mathscr{P})$ simple, the pairing

$$H_1(B, \iota_* \Lambda) \otimes H^1(B, \iota_* \check{\Lambda}) \to \mathbb{Z}$$

is perfect over $\mathbb{Q}$.

The induced map $H_1(B, \iota_* \Lambda) \to H^1(B, \iota_* \check{\Lambda})^* = H$, $\beta_{\mathrm{trop}} \mapsto \beta^*$ computes the period integrals:

$$\beta_{\mathrm{trop}} \mapsto \int_\beta \Omega = z^{\beta^*} t^{\langle c_1(\varphi), \beta^* \rangle} \in \mathbb{C}[H][t]$$