A QUICK INTRODUCTION TO STABILITY CONDITIONS

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Brief description of the lectures

In these lectures we give a crash course on *(Bridgeland) stability conditions*. In particular, after discussing the general results concerning any triangulated category, we will focus on the geometric case of $\mathrm{D}^b(X)$, the bounded derived category of coherent sheaves on a smooth projective variety.

During the three lectures, we expect to be able to cover the following topics:

- The topology of the space parametrizing stability conditions ([4]), focusing on some examples in dimension 1 and 2 (see [3, 10]);
- The geometric meaning of the so called *support property* and the wall and chamber structure ([2, 9]);
- The known geometric results in dimension 3, with an emphasis on the trivial canonical bundle case ([1, 2]).

We assume some familiarity with the following topics:

- Triangulated categories and derived categories of abelian categories (see [5] or, more extensively, the first three chapters of [7]), exact and Fourier–Mukai functors (see, for example, [6]);
- Slope stability of sheaves (see, for example, the first chapter of [8]).

1. Exercises

The following exercises assume the material discussed during the first lecture.

Exercise 1.1. Let X be a point. Describe $\mathrm{Stab}(X)$ and show that the support property is trivially satisfied in this example.

Exercise 1.2. Let X be a smooth complex projective curve of genus $g \geq 1$ and consider the pair $\sigma := (Z, \mathbf{A})$, where $\mathbf{A} = \mathbf{Coh}(X)$ and

$$Z\colon N(X) \to \mathbb{C} \qquad E \to -\deg(E) + \sqrt{-1}\,\mathrm{rk}(E).$$

Show that σ is a stability condition on $\mathrm{D}^b(X)$.

Exercise 1.3. Show that the two definitions of the support property are equivalent.

Exercise 1.4. Consider a t-structure $\left(\mathbf{T}^{\leq 0}, \mathbf{T}^{\geq 1}\right)$ on a triangulated category $\mathbf{T}$. Show that the inclusion

$$\mathbf{T}^{\leq 0} \hookrightarrow \mathbf{T} \qquad \text{and} \qquad \mathbf{T}^{\geq 1} \hookrightarrow \mathbf{T}$$

have right and left adjoints respectively.

References

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