TE-structures / ncHS from basic functions II 4.3.16

Thomas Reichelt

Recap $f: \mathbb{C}^\times \to \mathbb{C}$
$x \mapsto t = x + \frac{1}{x}$

$$\mathbb{C}[t] \to \underbrace{\mathcal{H}^0(f_+\mathcal{O}_{\mathbb{C}^\times})}_{} \xrightarrow{\widehat{loc}} \overbrace{\mathcal{D}_{\mathbb{C}}/(t^2-4)\partial_t^2 + t\partial_t}^{G \text{ from Claude's talk}} \to \mathbb{C}[t]$$

$$\simeq \mathbb{C}[t] \oplus \mathcal{D}_{\mathbb{C}}/(t^2-4)\partial_t + t$$

∂_t-localized

$$M' = \mathcal{D}_{\mathbb{C}}/(t^2-4)\partial_t + t \to G = \mathcal{D}_{\mathbb{C}}/(t^2-4)\partial_t^2 + t\partial_t$$

we conclude that $\widehat{loc}$ is actually the localization map from the exercise yesterday; localizing the sequence is exact functor and $1 \mapsto \partial_t$

$F^0 M$ generates M'

$$G_0^{(F)} = \sum_{j \geq 0} \partial_t^{-j}\, \widehat{loc}(F^0 M)$$

$z \mapsto \partial_t^{-1}$
$z^{-1} \mapsto \partial_t$
$z^2\partial_z \mapsto t$

$$G_0^{(F)} = \mathbb{C}[z] \langle z^2\partial_z \rangle \cdot z^{-1}$$

$$\subset \mathcal{D}/(z\partial_z - 1)(z\partial_z - 1) - 4z^{-2}$$

$$\downarrow z$$

$$\mathcal{D}/z^2(z\partial_z)(z\partial_z) - 4 \simeq G$$

$$G_0^{(F)} = \frac{\mathbb{C}[z]\langle z^2\partial_z\rangle}{z^2(z\partial_z)(z\partial_z) - 4}$$

Basis $1,\ z^2\partial_z$

$$z^2\partial_z \cdot 1 = z^2\partial_z$$
$$z^2\partial_z \cdot z^2\partial_z = z\, z^2\partial_z + 4$$

$$\rightsquigarrow \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix} \frac{dz}{z^2} + \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} \frac{dz}{z} \quad \text{TLE [as in Stoppa's talk]}$$

<u>General case:</u>

B $d \times n$ integer matrix $\quad T = (\mathbb{C}^\times)^d \quad \Lambda = \mathbb{C}^n$

$$\varphi_B : T \times \Lambda \to \mathbb{C} \times \Lambda$$

$$(t_1 .. t_d, \lambda_1 .. \lambda_n) \mapsto \left(\sum_i \lambda_i t^{b_i}, \lambda_1 .. \lambda_n\right)$$

$$t^{b_i} = \prod_{k=1}^{d} t_k^{b_{ki}}$$

$$\mathcal{H}^0(\varphi_{B+}\mathcal{O}_{T\times\Lambda})$$

$$A = \begin{pmatrix} 1 & 1 \dots & 1 \\ 0 & & \\ \vdots & \boxed{B} & \\ 0 & & \end{pmatrix} \quad (d+1)\times(n+1)$$

$$\mathbb{Z}A = \sum_i \mathbb{Z}a_i = \mathbb{Z}^{d+1} \quad \text{(columns)}$$

$$\mathbb{N}A = \mathbb{R}_{\geq 0}A \cap \mathbb{Z}^{d+1}$$

in previous example had $B = (1\ -1)$ $\qquad A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$

$\beta \in \mathbb{Z}^{d+1}$

$$\mathbb{C}^\times \times \mathbb{C}^2 \to \mathbb{C} \times \mathbb{C}^2$$
$$(x, \lambda_1, \lambda_2) \mapsto (\lambda_1 x + \lambda_2 \tfrac{1}{x}, \lambda_1, \lambda_2)$$

$I_A^B \subset \mathcal{D}_{\mathbb{C}^{n+1}}$ left ideal generated by

$$E_k = \sum_{i=0}^{n} a_{ki}\lambda_i\partial_i - \beta_k \quad \text{Euler operator}$$

$$\mathbb{L} = \{ l \in \mathbb{Z}^{d+1} \mid \sum_{i=0}^{n} l_i a_i = 0 \}$$

$$\Box_l = \prod_{l_i > 0} \partial_i^{l_i} - \prod_{l_i < 0} \partial_i^{-l_i} \quad \text{Box operator}$$

$l \in \mathbb{L}$ $\qquad$ GKZ system $\quad \mathcal{M}_A^\beta = \mathcal{D}_{\mathbb{C}^{n+1}}/I_A^\beta$

<u>Thm</u> Set $\gamma = \sum_{i=0}^{n} a_i$ $\quad \exists$ exact sequence

$$0 \to H^{d-1}(T) \otimes \mathcal{O}_{\mathbb{C}\times\Lambda} \to \mathcal{H}^0(f_+\mathcal{O}_{T\times\Lambda}) \xrightarrow{\widehat{loc}} \mathcal{M}_A^0 \to H^d(T) \otimes \mathcal{O}_{\mathbb{C}\times\Lambda} \to 0$$

$$F^p \mathcal{M}_A^0 = F^{ord}_{-p+d} \mathcal{M}_A^0$$

note that we talk about localizing at $z = \infty$, not $z = 0$ $\rightarrow$ this would remove transcendental data

<u>Cor:</u> $G = FL^{loc}\mathcal{H}^0(\varphi_B \mathcal{O}_{T\times\Lambda})$

$$= \frac{\Omega^d_{T\times\Lambda/\Lambda}[z^\pm]}{(zd - d\varphi\wedge)\Omega^{d-1}_{T\times\Lambda/\Lambda}} \simeq FL\, \mathcal{M}_A^0$$

$$\hat{E} = z^2\partial_z + \sum_{i=1}^{n} z\lambda_i\partial_i$$

$$\text{(12)}\quad \mathcal{D}/(\hat{\Box}, \hat{E})$$

$$\hat{\Box} = \prod_{l_i > 0} (z\partial_i)^{l_i} - \prod_{l_i < 0} (z\partial_i)^{-l_i}$$

$h: T \to \mathbb{P}^n$

$t_1 \dots t_d \mapsto (1 : t^{b_1} : \dots : t^{b_n}) = (w_0 : \dots : w_n)$

$X = \overline{\mathrm{im}(h)}$ $\qquad \mathbb{C}^n = \{w_0 \neq 0\}$

$Y = X \cap \{w_0 \neq 0\}$ $\qquad T \subset Y \subset X$

$D = X \setminus Y$

<u>Thm:</u> $U \subseteq \mathbb{C} \times \Lambda$ where the fibres have no singularities in $X \setminus Y$

$$G_0^{(F)} = \frac{\Omega^d_{Y\times\Lambda/\Lambda}(\log D)[z]}{(z\, d - d\varphi_B \wedge)\ \Omega^{d-1}_{Y\times\Lambda/\Lambda}(\log D)[z]}$$

$$= \frac{\mathbb{C}[z, \lambda_1 \dots \lambda_n] \langle z^2\partial_z, \partial_{\lambda_1} \dots \partial_{\lambda_n} \rangle}{(\hat{\Box}, \hat{E})}$$

Special case — like a fan spanning entire vector space

$\mathbb{N}B = \mathbb{Z}^d$ $\qquad Y = T$

$(G, G_0^{(F)})$ gives $\underset{\text{var.}}{\Lambda}$ pure ncHS on U

<u>Weight filtration</u>

$$\begin{array}{ccc} \mathcal{H}^0(\varphi_{B+}\mathcal{O}_{T\times\Lambda}) & \to & \mathcal{M}_A^0 \\ \uparrow & & \uparrow \partial_0 \dots \partial_n \\ \mathcal{H}^0(\varphi_{B\dagger}\mathcal{O}_{T\times\Lambda}) & \leftarrow & \mathcal{M}_A^{-\gamma} \end{array}$$

$$\begin{array}{ccc} T\times\Lambda & \xrightarrow{j} & Z \\ {\scriptstyle \varphi_B}\downarrow & \swarrow {\scriptstyle p} & \\ \mathbb{C}\times\Lambda & & \end{array}$$

$j_\dagger \to j_+$ plus, dagger

$W_{\min}\ \mathcal{M}_A^0$

fibres of φ_B compactify to hyperplane sections inside X

$\phi: Z_B \to \mathbb{C}\times\Lambda$ $\qquad Z_B$ singular as X typically singular

<u>Thm:</u> $W_{\min}\mathcal{H}^0(\varphi_{B+}\mathcal{O}_{T\times\Lambda}) \simeq \mathcal{H}^0(\phi_+ M^{IC}(Z_B))$

decomp. theorem $\simeq M^{IC}(Z', \mathcal{L}) \oplus IH^{d-1}(X) \otimes \mathcal{O}_{\mathbb{C}\times\Lambda}$

usually $Z' \simeq \mathbb{C}\times\Lambda$

fibre at λ $\mathcal{L} \simeq \mathrm{coker}\left[IH^{d-1}(X) \to IH^{d-1}(X \cap H_\lambda)\right]$

4.3.16

Stability & Stratifications

Vicky Hoskins

Let Q be a quiver and $d \in \mathbb{N}^V$ a dimension vector. Then for $\theta \in \mathbb{Z}^V$ and $\alpha \in \mathbb{Z}^V_{>0}$ the stratification on $\mathrm{Rep}_d Q$ agrees w. the GIT stratification.

Idea: HN types $\longrightarrow$ GIT index $\begin{pmatrix}\text{conj class of}\\ \text{1-PS}\end{pmatrix}$

$$\underbrace{\gamma = (d^1, \dots, d^r)}_{\text{dimension vectors}} \qquad \lambda_\gamma = (\lambda_{\gamma,v})_{v \in V}$$

$$\lambda_{\gamma,v}(t) = \begin{pmatrix} t^{s_1} I_{d_v^1} & & \\ & \ddots & \\ & & t^{s_r} I_{d_v^r} \end{pmatrix}$$

$$s_i = -\frac{\theta(d^i)}{\alpha(d^i)}$$

- King: lowest strata agree
- Inductively use this + description of strata

§ 4 Coherent sheaves

$(X, \mathcal{O}(1))$ proj var with $\begin{matrix}\mathcal{O}(1)\\ \downarrow\\ X\end{matrix}$ ample

Moduli: Classify coherent sheaves over X w fixed Hilbert polynomial P up to isomorphism.

Def: A coh sheaf $\mathcal{E}$ on X is n-regular if $H^i(\mathcal{E}(n-i)) = 0$ for $i > 0$

Facts about n-regular sheaves

1) Any $\mathcal{E}$ on X is n-regular for $n \gg 0$
2) Any n-regular sheaf is m-regular $\forall m \geq n$
3) If $\mathcal{E}$ is n-regular then $H^i(\mathcal{E}(n)) = 0$ for $i > 0$ and the evaluation map $(\dim H^0(\mathcal{E}(n)) = P(\mathcal{E}, n))$

$$\mathrm{eval}: H^0(\mathcal{E}(n)) \otimes \mathcal{O}(-n) \to \mathcal{E} \quad \text{is surjective}$$

Lemma: Every n-regular coh sheaf over X with Hilbert poly P is parametrised by an open subscheme of a Quot scheme

$$Q^{n\text{-reg}} \underset{\text{open}}{\subseteq} \mathrm{Quot}_n := \mathrm{Quot}_X(\mathbb{C}^{P(n)} \otimes \mathcal{O}(-n), P)$$

$$\| $$

$$\{ q: \mathbb{C}^{P(n)} \otimes \mathcal{O}(-n) \to \mathcal{E} \ : \ \mathcal{E} \text{ } n\text{-regular}, H^0(q(n)) \text{ is an isom.} \}$$

Moreover, $GL_{P(n)} \curvearrowright \mathrm{Quot}_n$ s.t.

$$\{ GL_n\text{-orbits in } Q^{n\text{-reg}} \} \overset{1:1}{\longleftrightarrow} \begin{Bmatrix}\text{iso classes of}\\ n\text{-reg coh sheaves}\\ \text{on } X \text{ w H poly } P\end{Bmatrix}$$

Proof: If $\mathcal{E}$ is n-regular w Hilbert poly P

$$\begin{array}{ccc} H^0(\mathcal{E}(n)) \otimes \mathcal{O}(-n) & \twoheadrightarrow & \mathcal{E} \\ (*) \wr\| & \nearrow & \\ \mathbb{C}^{P(n)} \otimes \mathcal{O}(-n) & & \end{array}$$

— point in $Q^{n\text{-reg}}$

The action of $GL_{P(n)}$ account for the choice of isomorphism $(*)$ ▨

§ 5 Semistability for sheaves

There are 2 notions for semistability

1) Slope semistability for torsion free sheaves
2) reduced Hilbert polynomial semistability for pure sheaves

(support of subsheaves = support of sheaf; Hilbert poly of s... same deg.)

Def (Rudakov)

A coh sheaf $\mathcal{E}$ on X is semistable if $\forall\, 0 \neq \mathcal{E}' \subseteq \mathcal{E}$ we have

$$\frac{P(\mathcal{E}', n)}{P(\mathcal{E}', m)} \leq \frac{P(\mathcal{E}, n)}{P(\mathcal{E}, m)} \quad \text{for all } m \gg n \gg 0$$

Notation /

$P(\mathcal{E}') \leq P(\mathcal{E})$ $[*]$

Boundedness theorem (Le Potier-Simpson)

There exists N st. $\forall n \geq N$ every semistable sheaf on X w Hilbert poly P is n-regular

$$\rightsquigarrow Q_n^{ss} \subseteq Q^{reg}$$

$GL_{P(n)} \nearrow$ $\qquad \|$

$$\{ q: \mathbb{C}^{P(n)} \otimes \mathcal{O}(-n) \twoheadrightarrow \mathcal{E} \ : \ \mathcal{E} \text{ semistable}$$

Let R be the closure of Q_n^{ss} in Quot_n

Thm (Seshadri, Gieseker, Maruyama, Simpson)

$$M^{ss}(X, P) = R /\!\!/_{L_{n,m}} GL_{P(n)} \quad \text{for } m \gg n \gg 0$$

For $m \gg n$

$$\mathrm{Quot}_n = \mathrm{Quot}(\mathbb{C}^{P(n)} \otimes \mathcal{O}(-n), P) \hookrightarrow \mathrm{Gr}(W, P(m))$$

$$\left(q : \mathbb{C}^{P(n)} \otimes \mathcal{O}(-n) \twoheadrightarrow \mathcal{E}\right) \mapsto \underbrace{H^0(q(m)) : W \twoheadrightarrow H^0(\mathcal{E}(m))}_{\dim P(m)}$$

choose m s.t. surjective

where $W = \mathbb{C}^{P(n)} \otimes H^0\mathcal{O}(m-n)$

Plücker

$L_{m,n} \xleftarrow{\text{pullback}} \mathcal{O}(1) \longrightarrow \mathbb{P}^N$

§6 Harder-Narasimhan stratifications

Lemma: Every coherent sheaf $\mathcal{E}$ on X has a! Harder-Narasimhan filtration

$$0 = \mathcal{E}^{(0)} \subseteq \mathcal{E}^{(1)} \ldots \subseteq \mathcal{E}^{(r)} = \mathcal{E}$$

s.t. $\mathcal{E}^{(i)} = \mathcal{E}^{(i)}/\mathcal{E}^{(i-1)}$ are semistable

and $P(\mathcal{E}^1) \succ \ldots \succ P(\mathcal{E}^s)$

see [*]

Def: The HN type of $\mathcal{E}$ is $T(\mathcal{E}) = (P(\mathcal{E}^1) \ldots P(\mathcal{E}^s))$

Rmk: There are inf many different HN types

eg $X = \mathbb{P}^1$ $P(x) = 2(x+1)$ rk 2, deg 0

Then $\mathcal{E}_n = \mathcal{O}(n) \oplus \mathcal{O}(-n)$ has HN type

$$T_n = (x+n+1, x-n+1)$$

Thm (Shatz, Nitsure)

Let $\mathcal{F}$ be a family over S of coherent sheaves on X w. Hilb poly P.

Then $S \longmapsto HNT$ is upper-semi continuous

$s \longmapsto T(\mathcal{F}_s)$

Hence there is a stratification $S = \coprod_\tau S_\tau$ into locally closed subschemes

Apply this to the universal quotient sheaf over the Quot scheme $\begin{matrix} \mathcal{U}_n \\ \downarrow \\ Q^{n\text{-reg}} \times X \end{matrix} \rightsquigarrow Q^{n\text{-reg}} = \coprod_\tau Q_\tau^{\text{reg}}$

HN stratification

§7 Comparison results for the stratifications

Let $Q^{n\text{-reg}} = \coprod_{\beta \in \mathcal{B}_{n,m}} S_\beta^{n,m}$ be the GIT instability stratification associated w the $GL_{P(n)}$ action on Quot_n restricted to $Q^{n\text{-reg}} \subseteq \mathrm{Quot}_n$

Q: What is the GIT index corresponding to a HN type?

Def: Let $\nu = (P_1 \ldots P_r)$ s.t. $\sum_{i=1}^{r} P_i = P$ then for each (m,n) define a GIT index $\beta_{n,m}(\nu) = ([\lambda], d)$ by

$$\lambda(t) = \begin{pmatrix} t^{s_1} I_{P_1(n)} & & \\ & \ddots & \\ & & t^{s_r} I_{P_r(n)} \end{pmatrix} \in GL_{P(n)}$$

where $s_i = \dfrac{P(m)}{P(n)} - \dfrac{P_i(m)}{P_i(n)}$ $\quad d = -\|\lambda\|$

Remark: If ν is a HN type then for $m \gg n \gg 0$ $\quad s_1 > \ldots > s_r$

Thm 1

Let T be a HN type then for $m \gg n \gg 0$ we have $Q_T^{n\text{-reg}} \subseteq S^{n,m}_{\beta_{n,m}(T)}$

HN stratum — closed subscheme — GIT stratum

Q: Why don't these stratifications agree?

1) the assignment HN types $\to$ GIT indices for fixed n,m is not injective unless $\dim X = 1$

$\nu \longmapsto \beta_{n,m}(\nu)$

2) For each T, we need to pick $m \gg n \gg 0$ depending on T

But there are infinitely many HN types!

<u>Moral reason:</u> The GIT instability stratification is finite but the HN stratification should be infinite

<u>Solution</u>

1) Refine the GIT stratification by collecting connected components

2) Construct an asymptotic GIT stratification using Quot_n for all n
$$\begin{array}{c}\mathrm{Quot}_n \\ \cup \\ Q^{n\text{-reg}}\end{array}$$

Recall every n-regular sheaf is n'-regular for $n' \geq n$

$$Q^{n\text{-reg}} \longrightarrow Q^{n'\text{-reg}}$$

$$\rightsquigarrow \left[Q^{n\text{-reg}}/GL_{P(n)}\right] \hookrightarrow \left[Q^{n'\text{-reg}}/GL_{P(n)}\right]$$

$$\overset{\shortparallel}{\mathrm{Coh}^{n\text{-reg}}_{X,P}}$$ stack of n-reg sheaves

The limit of this diagram is $\mathrm{Coh}_{X,P} = \bigcup_{n \geq 0} \mathrm{Coh}^{n\text{-reg}}_{X,P}$

The GIT and HN strata are preserved by $GL_{P(n)}$, so we get stratifications

GIT $\mathrm{Coh}^{n\text{-reg}} = \coprod_{\beta} S^n_\beta$ HN $\mathrm{Coh}^{n\text{-reg}}_{..} = \coprod_{\tau} \mathrm{Coh}^{\text{reg}}_{\tau}$

The HN strata are compatible

$$\rightsquigarrow \mathrm{Coh} = \coprod_{\tau} \mathrm{Coh}_\tau$$

For the GIT strata, we construct an asymptotic stratification as follows $n' > n$

$$\begin{tikzcd} S^{n,n'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & S^{n'}_\beta \arrow[d, hook] \\ S^n_\beta \arrow[r, hook] & \mathrm{Coh}^{n'\text{-reg}} \end{tikzcd}$$

<u>Thm 2</u>

1) For $n' \gg n \gg 0$ the $S^{n,n'}_\beta$ stabilize to S_β

2) $S_\beta \neq \emptyset \iff \beta(\tau) = \beta_{n,m}(\tau))$

HN type

3) $\mathrm{Coh}_\tau \simeq S_{\beta(\tau)}$

In particular on $\mathrm{Coh}_{X,P}$ the HN ~~filtration~~ stratifications with the asymptotic GIT stratifications