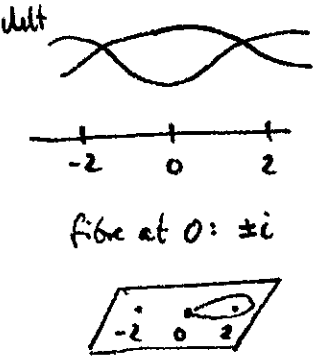
TE-structures / ncHS from tame fuchs 3.3.16

Thomas Reichelt



$f : \mathbb{C}^* \to \mathbb{C}$

$x \longmapsto t = x + \frac{1}{x}$

$x - t - \frac{1}{x} = 0$

$x = \frac{1}{2}\left(t \pm \sqrt{t^2-4}\right)$

$$H^0(f^{-1}(t), \mathbb{Q}) = \begin{cases} \mathbb{Q}^2 & t \neq \pm 2 \\ \mathbb{Q} & t = \pm 2 \end{cases}$$ vanishing cycle

$R f_* \underline{\mathbb{Q}}_{\mathbb{C}^*}$

$R^k f_* \mathbb{Q} = 0 \quad k \neq 0$

$R^0 f_* \mathbb{Q}|_{\mathbb{C}\setminus\{\pm 2\}}$ local system of rk 2

$$H^0(f^{-1}(0), \mathbb{Q}) = \underset{e_1}{H^0(\{+i\}, \mathbb{Q})} \oplus \underset{e_2}{H^0(\{-i\}, \mathbb{Q})}$$

$e_1 + e_2$ monodromy invariant

$e_1 - e_2$ vanishing at ± 2

$R^0 f_* \mathbb{Q}|_{\mathbb{C}\setminus\{\pm 2\}} \simeq \underline{\mathbb{Q}} \oplus \mathcal{L}_{\nwarrow \text{rk } 1}$

$R^0 f_* \mathbb{Q} \simeq \underline{\mathbb{Q}} \oplus j_* \mathcal{L}$

D-module picture

$D_{\mathbb{C}} = \mathbb{C}[t]\langle \partial_t \rangle$ Weyl Algebra $\quad \partial_t t - t\partial_t = 1$

$D_{\mathbb{C}}/\partial_t \simeq \mathbb{C}[t] \dashrightarrow \underline{\mathbb{C}}$

$D_{\mathbb{C}}/(\partial_t t) \simeq \mathbb{C}[t, t^{-1}] \dashrightarrow R j_* \underline{\mathbb{C}}$

$D_{\mathbb{C}}/(t\partial_t - \alpha) \dashrightarrow j_* \mathcal{L}$

$\mathcal{L}$ rk mond. $e^{2\pi i\alpha}$, $\alpha \notin \mathbb{Z}$

reg. hol. D-mod $\xleftrightarrow{\text{Riemann Hilbert}}$ perverse sheaves 2-term complexes

reg. holonomic means every formal sol. is analytic

$R f_* \mathbb{Q}$

1/2

$f_+ \mathcal{O}_{\mathbb{C}^\times} = \left(\Omega^0_{\mathbb{C}^\times}[\partial_t] \to \Omega^1_{\mathbb{C}^\times}[\partial_t] \right)[1]$

$g \otimes \partial_t^n \mapsto dg \otimes \partial_t^n - g\,df \otimes \partial_t^{n+1}$

$x^k \otimes \partial_t^n \mapsto k x^k \frac{dx}{x} \partial_t^n - x^k\left(x - \frac{1}{x}\right) \frac{dx}{x} \otimes \partial_t^{n+1}$

$$\mathcal{H}^0(f_+ \mathcal{O}_{\mathbb{C}^\times}) = \frac{\Omega^1_{\mathbb{C}^\times}[\partial_t]}{(d - df \wedge \partial_t)\Omega^0_{\mathbb{C}^\times}[\partial_t]}$$

$\partial_t(\omega \otimes \partial_t^n) = \omega \otimes \partial_t^{n+1}$

$t \cdot (\omega \otimes \partial_t^n) = \omega f \otimes \partial_t^n - n\omega \otimes \partial_t^{n-1}$

Decomposition of $\mathcal{H}^0(f_+ \mathcal{O}_{\mathbb{C}^\times})$

$\partial_t\left(-\left(x - \frac{1}{x}\right) \frac{dx}{x} \otimes \partial_t^0\right) = -\left(x - \frac{1}{x}\right) \frac{dx}{x} \otimes \partial_t$

$= (d - df \wedge \partial_t)(1 \otimes \partial_t^0) = 0$ in $\mathcal{H}^0(f_+ \mathcal{O}_{\mathbb{C}^\times})$

$\frac{dx}{x} \otimes \partial_t^0$ is annihilated by $(t^2-4)\partial_t + t$

$(t-2)(t+2)$

$\overset{\shortparallel}{y}$

$y(y+4)\partial_y + y + 2$

$y\partial_y + \frac{y+2}{y+4} = \frac{1}{2} + \mathcal{O}(y^2)$

$$\begin{tikzcd} \mathcal{H}^0(f_+ \mathcal{O}_{\mathbb{C}^\times}) \simeq \mathbb{C}[t] \oplus D_{\mathbb{C}}/((t^2-4)\partial_t + t) \arrow[d, bend right, "RH"'] \\ R^0 f_* \mathbb{Q} \simeq \mathbb{Q} \oplus j_* \mathcal{L} \end{tikzcd}$$

$\left(\mathcal{H}^0(f_+ \mathcal{O}_{\mathbb{C}^\times}), R^0 f_* \mathbb{Q}\right) \rightsquigarrow$ mixed Hodge module

F and W are filtrations by sub-D-mod. and sub-perverse sheaves

W here is trivial

Hodge filtration on $\mathcal{H}^0(f_+ \mathcal{O}_{\mathbb{C}^\times})$ is a filtration by coherent $\mathcal{O}_{\mathbb{C}}$-modules

$H^0(\mathbb{C}^\times) \longrightarrow H^0(f^{-1}(0),\mathbb{C})$

weight 0

$F^0H^0(\mathbb{C}^\times) = H^0(\mathbb{C}^\times)$

$F^1H^0(\mathbb{C}^\times) = 0$

$H^0(\mathbb{C}^\times)\otimes\mathbb{C}[t] \longrightarrow \mathcal{H}^0 f_+\mathcal{O}_{\mathbb{C}^\times} \simeq \underbrace{\mathbb{C}[t] + D_{\mathbb{C}}/(t^2-4)\partial_t+t}_{M \text{ in Claude's notation}}$

$1\in\mathbb{C}[t]\subset F^0\mathcal{H}^0(f_+\mathcal{O}_{\mathbb{C}^\times})$

$\mathbb{C}(t)\subset F^0\mathcal{H}^0(f_+\mathcal{O}_{\mathbb{C}^\times})$

$1\in D_{\mathbb{C}}/(t^2-4)\partial_t+t$

$1\in F^0\mathcal{H}^0(f_+\mathcal{O}_{\mathbb{C}^\times})$

$\partial_t^p\in D_{\mathbb{C}}/((t^2-4)\partial_t+t$

$\partial_t^p\in F^{-p}\mathcal{H}^0(f_+\mathcal{O}_{\mathbb{C}^\times})$

$F^{ord}_\bullet D_{\mathbb{C}}$ $\quad F^{ord}_0 D_{\mathbb{C}} = \mathbb{C}[t]$

$F^{ord}_1 D_{\mathbb{C}} = \mathbb{C}[t]+\mathbb{C}[t]\partial_t$

$F^{ord}_p D_{\mathbb{C}}/(t^2-4)\partial_t+t = F^{-p}D_{\mathbb{C}}/((t^2-4)\partial_t+t$

Fourier-Laplace trafo

$G = \mathbb{C}[t]\langle\partial_t,\partial_t^{-1}\rangle \otimes_{\mathbb{C}[t]\langle\partial_t\rangle} M$

G is a $\mathbb{C}[z,z^{-1}]\langle\partial_z\rangle$

$z\cdot m = \partial_t^{-1}$

$z^{-1}\cdot m = \partial_t m$

$z^2\partial_z m = tm$

for cyclic D-modules like $D_{\mathbb{C}}/(t^2-4)\partial_t+t$ it is easy to compute FL-trafo

Problem in general it is hard to compute M as a cyclic $D_{\mathbb{C}}$-module

$T=\mathbb{C}^\times$

$0\to H^0(\mathbb{C}^\times)\to H^0(f^{-1}(t))\to H^1(T,f^{-1}(t))$

$\to H^1(\mathbb{C}^\times)\to 0$

$H^0\ \mathbb{C}[t]\to\mathcal{H}^0 f_+\mathcal{O}_{\mathbb{C}^\times}$?

Compute $H^1(T,f^{-1}(t))\simeq H_1^{BM}(T,f^{-1}(t))$ in terms of Borel-Moore homology

$H_0^{BM}(f^{-1}(t))$ $\quad H_1^{BM}(T,f^{-1}(t))$ $\quad H_1^{BM}(T)$

$\mathbb{C}[t]\to\mathbb{C}[t]\oplus D_{\mathbb{C}}/(t^2-4)\partial_t+t \to D_{\mathbb{C}}/(t^2-4)\partial_t^2+t\partial_t \to \mathbb{C}$

$D_{\mathbb{C}}/\partial_t \cong \mathbb{C}[t]$

$1/F_0 \longrightarrow \partial_t$

$\in F_0$

Introduction to nc Hodge structures IV

3.3.16

Claude Sabbah

Example of pure pol ncHS

Ex. of pure pol HS $H = \overset{\perp}{\bigoplus_p} H^{p,w-p}$ $h>0$

vw ---- " ---- on $\mathbb{A}^1 \setminus C$

$\pi_1(\mathbb{A}^1 \setminus C, *) \longrightarrow U_\mu(\mathbb{C})$ unitary group

n $T_1, \dots T_n$ get a unitary loc. sys. of rk μ on $\mathbb{A}^1 \setminus C$

$w = 0$

$H = H$ trivial decomposition

(V, ∇) $F^pV = \begin{cases} V & p \leq 0 \\ 0 & p > 0 \end{cases}$

Top data for an ncHS

$(\mathcal{H}, \nabla)$ free $\mathbb{C}\{z\}$-mod ∇ | $(\mathcal{L}, \mathcal{L}_\cdot)$ over $\mathbb{Q}$ $\mathcal{Q}$

Data: C finite set $\subset \mathbb{C}$ $\# C = n$

- matrix Σ_1 size $d \geq 1$ block upper triangular, n blocks
- $\theta_0 \in S^1$ generic wrt C $\rightsquigarrow$ total order on $C = \{c_1 .. c_n\}$
- diag blocks of Σ_1 $\Sigma_{c_1} \dots \Sigma_{c_n}$
- $w \in \mathbb{Z}$

Goal of today: produce $\text{ncHS}(C, \Sigma_1, \theta_0, w)$ (base point ↓ θ_0)

Replace $(\mathcal{L}, \mathcal{L}_\cdot)$ in terms of $\Sigma_1 \Leftrightarrow$ endom of $\mathbb{Q}$

$$\bigoplus_{c \in C} L_{1,c} \underset{S'}{\overset{S}{\rightleftarrows}} \bigoplus_{c \in C} L_{2,c}$$

S upper triangular S' lower triangular

$$\mathbb{Q}^d \simeq \bigoplus_c \mathbb{Q}^{d_c} \underset{\Sigma'}{\overset{\Sigma_1}{\rightrightarrows}} \bigoplus \mathbb{Q}^{d_c}$$

L_1 L_2

$S' : (-1)^w \, {}^t\Sigma_1 = \Sigma'$

Pairing $\mathcal{Q} : (\mathcal{L}, \mathcal{L}_\cdot) \otimes (\mathcal{L}, \mathcal{L}_\cdot) \to (\underline{\mathbb{Q}}, \underline{\mathbb{Q}}_\cdot)$

$$\mathcal{Q} \Leftrightarrow \boxed{\begin{array}{l} Q_{12} : L_1 \otimes L_2 \to \mathbb{Q} \\ Q_{2,1} : L_2 \otimes L_1 \to \mathbb{Q} \\ Q_{2,1}(x_2, x_1) = Q_{12}(S^{-1}x_2, S'x_1) \end{array}}$$

$\mathcal{Q}$ is $(-1)^w$-symmetric

$\Leftrightarrow Q_{21}(x_2, x_1) = (-1)^w Q_{12}(x_1, x_2)$

$(\mathcal{L}, \mathcal{L}_\cdot) \otimes_{\mathbb{Q}} \mathbb{C} \leftrightarrow (\mathcal{G}, \nabla)$ localized flat bdl

localized flat bdl $(\mathbb{C}(\{z\}), \nabla)$

$$\mathbb{C}((z)) \otimes \mathcal{G} = (\hat{\mathcal{G}}, \nabla) \simeq \bigoplus_{c \in C} \Big(\hat{\mathcal{G}}_c, \nabla_c + \overset{\text{reg.}}{\downarrow} c\frac{dz}{z^2}$$

$(\hat{\mathcal{G}}_c, \nabla_c)$ reg sing cong ct $\boxed{\mathcal{G}_c, \nabla_c}$ having reg. sing

Deligne-Malgrange lattices

$\mathbb{C}\{z\}$-mod in a $\mathbb{C}(\{z\})$-vect. sp. which generates it

on each $V^\alpha \mathcal{G}_c$ free $\mathbb{C}\{z\}$-mod on which ∇ has a simple pole

$\text{Re}(\text{eigenval of Res}\,\nabla \text{ in } V^\alpha \mathcal{G}_c) \in [\alpha, \alpha+1)$

Fix $\alpha\in\mathbb{R}$

$$V^\alpha \hat{\mathcal{G}}_c = \mathbb{C}[[z]] \otimes_{\mathbb{C}\{z\}} V^\alpha \mathcal{G}_c \subset \hat{\mathcal{G}}_c$$

$$V^\alpha \hat{\mathcal{G}} = \bigoplus_{c\in C} V^\alpha \hat{\mathcal{G}}_c \subset \hat{\mathcal{G}}$$

Lemma (Malgrange) in general finding $(\mathcal{H},\nabla)\subset(\mathcal{G},\nabla)$ is eq. to finding lattices $(\hat{\mathcal{H}},\nabla)\subset(\hat{\mathcal{G}},\nabla)$

If $\hat{\mathcal{H}}$ is known, define $\mathcal{H}$ as $\hat{\mathcal{H}}\cap\mathcal{G}$ in $\hat{\mathcal{G}}$

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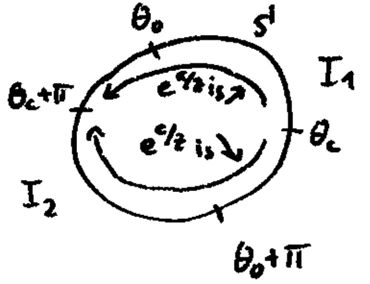
how to get from Stokes filtered local system to stokes data?



$\mathcal{L}$ loc.sys.

θ_0

on each $(\theta_0-\varepsilon, \theta_0+\pi+\varepsilon)$

and $(\theta_0+\pi-\varepsilon, \theta_0+2\pi+\varepsilon)$

$\mathcal{L}_{\le c}$

$$\begin{tikzcd} L_1=\Gamma(I_1,\mathcal{L}) \arrow[d, bend right, "S_{\theta_0}"'] \arrow[d, bend left, "S'_{\theta_0+\pi}"] \\ L_2=\Gamma(I_2,\mathcal{L}) \end{tikzcd}$$

∃! one Stokes line wrt any $(c,c')\in C^2$

$(\mathcal{G},\nabla)$ $(\mathcal{G},\nabla)^\vee$ dual connection

$$[DM^\alpha(\mathcal{G})]^\vee \simeq DM^{>-\alpha-1}(\mathcal{G}^\vee)$$

$$[DM^{>\alpha}(\mathcal{G})]^\vee \simeq DM^{-\alpha-1}(\mathcal{G}^\vee)$$

Cor: $Q:(\mathcal{G},\nabla)\otimes(\mathcal{G},\nabla)\to(\mathbb{C}(\{z\}),d)$

$\Leftrightarrow Q_B:(\mathcal{L},\mathcal{L}_\bullet)\otimes\iota^{-1}(\mathcal{L},\mathcal{L}_\bullet)\to(\mathbb{C},\mathbb{C}_\bullet)$ non-deg pairing

then $Q: DM^\alpha(\mathcal{G},\nabla)\otimes\iota^* DM^{>-\alpha-1}(\mathcal{G},\nabla)$

$$\downarrow \text{non-deg.}$$

$$(\mathbb{C}\{z\},0)$$

Cor: Take $\alpha\in\mathbb{Z}$

1) If none of the mon $\hat{\mathcal{G}}_c$ has 1 as an eigenval

$(-1)^w \Sigma_{1c}^{-1}\,{}^t\Sigma_c$

↑ block diag part corresp to $c\in C$

then $DM^\alpha\otimes\iota^* DM^\alpha \xrightarrow{\text{nondeg}} (z^{2\alpha+1}\mathbb{C}\{z\},d)$

$(DM^\beta = DM^{>\beta}\ \forall\beta\in\mathbb{Z})$

2) If -1 is not an eigenval

$$DM^{\alpha-1/2}\otimes i^* DM^{\alpha-1/2}\to(z^{2\alpha}\mathbb{C}\{z\},d)$$

Thm: Assume

1) $\forall c\ \ \ker(\Sigma_{1c}+{}^t\Sigma_{1c})=0$

2) $\Sigma_1+{}^t\Sigma_1$ is pos. semi definite sym. matrix

$\Downarrow$

$(DM^{-(w+1)/2},(\mathcal{L},\mathcal{L}_\bullet),Q,w)$

is a pol. ncHS

Stability & Stratifications

Victoria Hoskins

§1 Summary

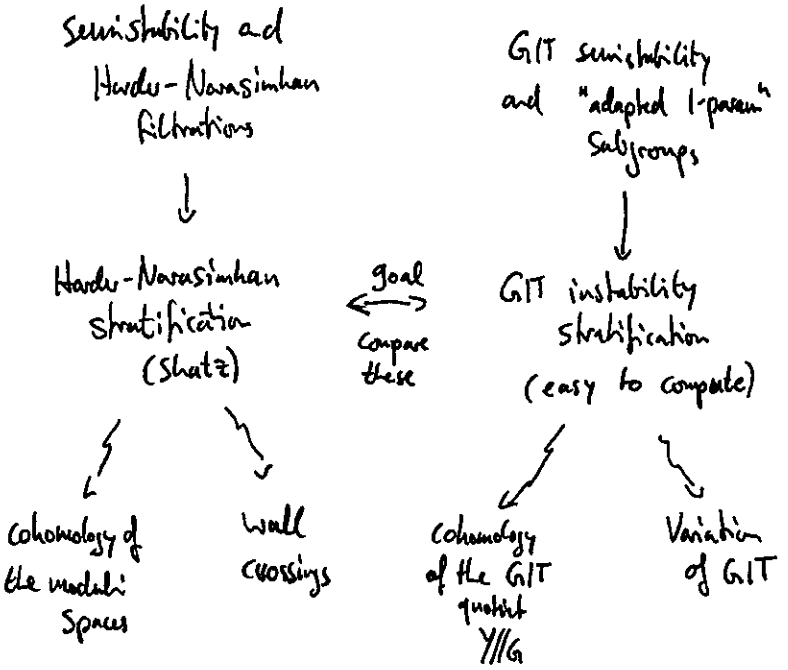
- Moduli problems
  1) quiver representations
  2) coherent sheaves on a proj. variety

Geometric invariant theory

$G \curvearrowright Y$ proj. or affine variety, reductive group

GIT quotient

$Y^{ss} \to Y/\!/G$



§2 Geometric Invariant Theory

Motivation: construct moduli space as group quotients in algebraic geometry

Affine GIT

$G \curvearrowright X = \mathrm{Spec} A$

linear alg. grp

$G \curvearrowright \mathcal{O}(X) = A$ by $(g \cdot f)(x) = f(g^{-1} \cdot x)$

Thm (Nagata, Hilbert)
If G is reductive $\mathcal{O}(X)^G$ is finitely generated

Def: G is reductive $\iff G = K_{\mathbb{C}}$ for $K \leq G$ max cpt
(over $\mathbb{C}$)

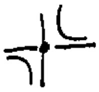
Eg $G = GL_n \geq K = U(n)$ is reductive

$G = \mathbb{G}_a = (\mathbb{C}, +) \geq K = \{0\}$ is not reductive

Thm (Hilbert, Mumford)
If G is reductive, $\pi : X \to X/\!/G := \mathrm{Spec}\,\mathcal{O}(X)^G$ is a categorical quotient

Examples $G = \mathbb{G}_m \curvearrowright X = \mathbb{A}^2$

1) $t \cdot (x,y) = (t^{-1}x, ty)$



$\mathcal{O}(X)^G = \mathbb{C}[xy]$

$X \to X/\!/G \cong \mathbb{A}^1$

2) $t \cdot (x,y) = (tx, ty)$

$\mathcal{O}(X)^G = \mathbb{C}$

$X \to X/\!/G = *$

Mumfords GIT w. linearizations

Assume G is reductive and either

1) $G \overset{\text{linear}}{\curvearrowright} Y \underset{\text{affine}}{\subseteq} \mathbb{A}^n$ and $\mathcal{L} = Y \times \mathbb{A}^1$ has a G-action give by $\chi : G \to \mathbb{G}_m$

$g \cdot (y,z) = (g \cdot y, \chi(g) z)$

or

2) $G \curvearrowright Y \overset{\text{|X|}}{\subseteq} \mathbb{P}^n$ G-equivariant, $\mathcal{L} \downarrow Y$ ample line bundle

Idea
Replace $\mathcal{O}(X)^G$ by $\bigoplus_{r \geq 0} H^0(Y, \mathcal{L}^{\otimes r})^G$

Def: $Y \dashrightarrow \mathrm{Proj}(\quad) = Y/\!/_{\mathcal{L}} G$

open $\cup$ $Y^{ss}(\mathcal{L}) \nearrow$ GIT quotient

locus on which this rat^l map is defined

Thm (Mumford)
$Y^{ss}(\mathcal{L}) \to Y/\!/_{\mathcal{L}} G$ is a categorical quotient

Hilbert-Mumford Criterion

$y \in Y$ is $\mathcal{L}$-semistable $\iff \mu^{\mathcal{L}}(y,\lambda) \geq 0$ $\forall$ 1 param subgrps $\lambda : \mathbb{G}_m \to G$ s.t. $\underset{\parallel\, y_0}{\lim_{t \to 0} \lambda(t) \cdot y}$ exists

where
$\mu^{\mathcal{L}}(y,\lambda)$ = wt of the $\mathbb{G}_m$-action on $\mathcal{L}_{y_0}$
$(= \langle \chi, \lambda \rangle$ in case 1)

Rmk $\mu^{\mathcal{L}}(y, \lambda^n) = n \mu^{\mathcal{L}}(y, \lambda)$ for $n \in \mathbb{Z}_{\geq 0}$

Idea: Stratify $Y - Y^{ss}(\mathcal{L}) = Y^{us}(\mathcal{L})$ using a normalised Hilbert-Mumford weight

Norm: Fix $T \subseteq G$ max^{l} torus and pick a Weyl invariant norm $\|-\|$ on $X_*(T)_{\mathbb{R}}$

$\lambda : \mathbb{G}_m \to G$

$g\lambda g^{-1} : \mathbb{G}_m \to T$ for some g $\quad \|\lambda\| := \|g\lambda g^{-1}\|$

Ex a) $G = GL_n \supseteq T = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$

$X_*(T) \cong \mathbb{R}^n \curvearrowleft W = S_n$

$\|-\|$ = Euclidean norm

b) $G = GL_{n_1} \times \dots \times GL_{n_r}$

Fix $\alpha \in \mathbb{Z}^r_{>0}$ then $\underset{\underset{(\lambda_1 .. \lambda_r)}{/\!/}}{\|\lambda\|^2_\alpha} = \sum_{j=1}^{r} \alpha_j \|\lambda_j\|^2_E$

Thm (Kempf, Hesselink, Kirwan)

Let $G \curvearrowright Y$ as in 1) or 2) above and fix a norm $\|-\|$ on the 1-P.S. of G then

$\exists$ stratification $Y = \coprod_{\beta \in \mathcal{B}} S_\beta$ into finitely many G-inv^t locally closed subschemes S_β

s.t.

- $\overline{S_\beta} \subseteq \coprod_{\gamma \geq \beta} S_\gamma$
- $\mathcal{B}$ can be determined from the weights of the action of a maximal torus
- S_β is determined from a simpler limit set which is a GIT ss set for a smaller red. grp action

Construction For $y \in Y^{us}(\mathcal{L})$

$$M^{\mathcal{L}}(y) = \inf \left\{ \frac{\mu^{\mathcal{L}}(y,\lambda)}{\|\lambda\|} : \begin{array}{l} \text{1-PS } \lambda : \mathbb{G}_m \to G \\ \text{s.t. } \lim_{t \to 0} \lambda(t) \cdot y \text{ exists} \end{array} \right\}$$

Def^n: A primitive 1-PS achieving this inf is said to be adapted to y $\quad \Lambda^{\mathcal{L}}_{\|\cdot\|}(y) = \{ \lambda \text{ adapted to } y \}$

2/3

Then $\beta = ([\lambda], d)$ $\in \mathbb{Q}_{<0}$

$S_\beta = \{ y \in Y : M^{\mathcal{L}}(y) = d \text{ and } [\lambda] \cap \Lambda^{\mathcal{L}}_{\|\cdot\|}(y) \neq \phi \}$

§ 3 Quiver representations

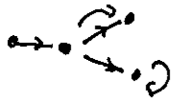
Let $Q = (V, A, h, t)$ be a quiver rep

vertex set V $\quad h, t : A \to V$

arrow set A

Def: A rep of Q is $W = (W_v, v \in V, f_a, a \in A)$ where W_v is a $\mathbb{C}$-vector space

$f_a : W_{t(a)} \to W_{h(a)}$



$\underline{\dim}(W) = (\dim W_v)_{v \in V}$

dim vector

Idea: Fix $d \in \mathbb{N}^V$, classify reps of Q of dimension d up to isomorphism

Lemma: On $Rep_d(Q) = \bigoplus_{a \in A} Hom(\mathbb{C}^{d_{t(a)}}, \mathbb{C}^{d_{h(a)}})$ there is an action of $G_d(Q) = \prod_{v \in V} GL_{d_v}$ s.t. that the orbits correspond to isom. classes of rep of Q of dim d.

Rmk: 1) Any coarse moduli space is a categorical quotient of this action

2) Affine GIT does not give a satisfactory quotient

$\mathcal{O}(Rep_d Q)^{G_d(Q)}$ is generated by traces of oriented cycles in Q (Le Bruyn-Procesi)

King's construction of moduli spaces

$\chi : G_d(Q) \to \mathbb{G}_m \iff \theta \in \mathbb{Z}^V$

$(g_v)_v \mapsto \prod_{v \in V} \det(g_v)^{\theta_v} \leftarrow\!\shortmid \theta$

Assume $\sum_v \theta_v d_v = 0$

Def: A d-dim^l rep W of Q is θ-semistable if $\forall\ 0 \neq W' \subseteq W$

$$\theta(W') := \sum_{v \in V} \theta \dim W'_v \geq 0$$

King: GIT ss for $\chi_\theta \Leftrightarrow \theta$-ss

$$M^{\theta ss}(Q,d) = \mathrm{Rep}_d Q /\!\!/_{\chi_\theta} G_d(Q) \quad \text{is}$$

a coarse moduli space for S-equivalence classes of θ-ss representations of Q.

Fix $\alpha \in \mathbb{Z}_{>0}^{V}$

<u>Def</u>: A representation W of Q is (θ,α)-semistable if $\forall\ 0 \neq W' \subseteq W$

$$\frac{\theta(W')}{\alpha(W')} \geqslant \frac{\theta(W)}{\alpha(W)}$$

<u>Exc</u>: Every rep W of Q has a unique HN filtr. wrt (θ,α)

$$0 = W^{(0)} \subsetneq W^{(1)} \subseteq \dots \subseteq W^{(r)} = W$$

s.t. $W^i = W^{(i)}/W^{(i-1)}$ are (θ,α)-ss. and $\mu_{\theta,\alpha}(W^1) < \dots < \mu_{\theta,\alpha}(W^r)$

<u>Thm (H.)</u>

The GIT stratif. for $G_d Q \curvearrowright \mathrm{Rep}_d Q$ linearised by χ_θ and using the norm $\|-\|_\alpha$ coincides with the stratification by HN types

$$\alpha(W) = \sum_{v \in V} \alpha_v \dim W_v$$

$$\left[\mathrm{Rep}_d^{\theta ss} Q / G_d(Q)\right] \subseteq \left[\mathrm{Rep}_d Q / G_d Q\right]$$

Stability on tropical vertex groups III

Jacopo Stoppa

$n = \mathrm{rk}\,\Gamma$

Γ lattice

$\langle -,- \rangle$

basis $\alpha_1, \alpha_2, \ldots \alpha_n \in \Gamma$

work w subalg. of $\mathfrak{g}_\Gamma$ gen^d by $e_{\alpha_1} \ldots e_{\alpha_n}$

$\hat{\mathfrak{g}}_\Gamma :=$ completion of algebra wrt $(e_{\alpha_1}, \ldots e_{\alpha_n})$

Basic symplectomorphisms $\Theta^{\Omega}_{1+e_\alpha}$ $\alpha \in \Gamma$, $\Omega \in \mathbb{Q}$

α will always be in the monoid spanned by $\alpha_1 \ldots \alpha_n$

Basic Symplectomorphisms

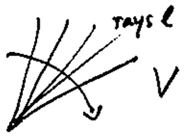
"Central charge" $Z \in \mathrm{Hom}(\Gamma, \mathbb{C})$

allows to define ordered products

$$(*) \quad \overrightarrow{\prod_{\ell \subset V}} \prod_{Z(\alpha) \in \ell} \Theta^{\Omega(\alpha)}_{1+e_\alpha}$$

$V \subset \mathbb{C}$ strictly convex cone

~~allows to define ordered products~~



Stability data

Function $\Omega = \Omega(\alpha, Z)$ s.t. the ordered product $\overrightarrow{\prod_{\ell \subset V}} \prod_{Z(\alpha) \in \ell} \Theta^{\Omega(\alpha, Z)}_{1+e_\alpha}$ is constant as long as no rays ℓ with a non-trivial factor enter or leave the cone V as Z varies in $U \subset \mathrm{Hom}(\Gamma, \mathbb{C})$

Rmk: $(*)$ implies $\Omega(\alpha, Z)$ is locally constant in Z it can jump

How do we get stability data?

1) First talk
2) DT invariant counting sstables in suitable categories produce stab. data (Joyce, KS)

Möbius transform:

$$DT(\alpha, Z) = \sum_{\substack{k > 0 \\ k | \alpha}} \frac{\Omega(\alpha/k, Z)}{k^2}$$

↗ Donaldson-Thomas invariant

Goal: Start from stability data, construct (family of) Irregular connections on $\mathbb{P}^1$

Shape of connections

$$\nabla(Z) = d - \left(\frac{A^{(-1)}}{z} + A^{(0)} + z A^{(1)} \right) \frac{dz}{z}$$

$$\left[\begin{array}{l} \text{compare to} \\ d - \left(\frac{U}{z} - Q - z U^* \right) \frac{dz}{z} \end{array} \right]$$

from ncHS

- $A^{(i)} \in D(\hat{\mathfrak{g}}_\Gamma) =$ comm alg derivations (n.nec. Poisson) they are connections on trivial principle $\mathrm{Aut}(\hat{\mathfrak{g}}_\Gamma)$-bdl on $\mathbb{P}^1$.
- $A^{(i)} = A^{(i)}(Z)$ are complicated functions of central charge but constant on $\mathbb{P}^1$
- The formal type of $\nabla(Z)$ at $z=0$ is $d - \frac{Z}{z^2}$ equivalence class under formal power series gauge transformations

$$\left[\text{rmk: } Z \text{ is a } \underset{\text{comm.alg.}}{\wedge} \text{derivation} \quad Z(e_\alpha) = Z(\alpha)\, e_\alpha \right]$$

- (anti-) Stokes rays at $z=0$ $\ell_\alpha(Z) = Z(\alpha)$ $\forall \alpha$ s.t. $\Omega(\alpha, Z) \neq 0$
- Stokes factor along the ray $\ell_\alpha(Z)$ is $\boxed{\prod_{Z(\alpha) \in \ell} \Theta^{\Omega(\alpha, Z)}_{1+e_\alpha}} =: S_{\ell_\alpha}$

(All this for generic central charges!)

[$X_{\alpha_1 \alpha_2}$ $X_{\alpha_2 \alpha_3}$ can. flat sections ← explain what Stokes factors are.

Problem: ∞ many Stokes rays (dense in some sector!)

But only finitely many mod $(e_{\alpha_1}, \ldots e_{\alpha_n})^p$ $p \geq 1$

We take the direct limit $p \to \infty$

Construction of $\nabla(Z)$: Riemann-Hilbert factorisation problem

Construct $X(z, Z): \mathbb{C}^\times \to \mathrm{Aut}(\hat{\mathfrak{g}}_\Gamma)$ s.t.

1) ~~They~~ It ~~are~~ is holomorphic in Stokes sectors
2) $X(z, Z)(e_\alpha)$ extends to holomorphic function on ray $\ell_\alpha(Z)$
3) Jump condition $z_0 \in \ell$
$$X(z_0^+, Z) = X(z_0^-, Z) \circ S_\ell$$
4) Boundary conditions at $z \to 0$ $z \to \infty$

Solve the RH problem by finding a fixed point of a suitable integral operator

$$\mathcal{Z}(\underbrace{Y(z,Z)}_{\mathrm{End}(\mathfrak{g})\text{-valued}})(e_\alpha) = e_\alpha X_0(z,Z)\exp_*\left(\sum_\beta{}' \Omega(\beta,Z)\langle\alpha,\beta\rangle \int_{\ell_\beta(Z)} \log(1+Y(z,Z)e_\beta)\right)$$

integral is wrt. measure

$$\rho(Z,z') = \frac{1}{2\pi i}\frac{z'+z}{z'-z}dz'$$

Base point function $X_0(z,Z)(e_\alpha) = \exp\left(-\frac{1}{z}Z(\alpha) - z\bar{Z}(\alpha)\right)e_\alpha$

<u>Exc:</u>
Lagrange inversion formula
this takes endom. to autom.

- A fixed point $X(z,Z)$ of $\mathcal{Z}$ would give a sol to RH problem and so the connection $\nabla(Z)$ essentially by Plemelji's theorem

Can rewrite $\mathcal{Z}$ in a simple way using DT:

$$\mathcal{Z}(Y)(e_\alpha) = e_\alpha X_0 \exp_*\left(\sum_\beta{}'\langle\alpha,\beta\rangle DT(\beta,Z)\int_{\ell_\beta}\frac{dz'}{z'}\rho(z,z')Y(z')(e_\beta)\right)$$

<u>Prop:</u>
Iteration of $\mathcal{Z}$ starting from $X_0(z,Z)$ gives a (solution) well defined fixed point.

<u>Prop:</u> There is an explicit formula for our flat section as a sum over graphs

$$X(z,Z)(e_\alpha) = X_0(z,Z)e_\alpha\exp_*\left(\langle\alpha, -\sum_T{}' W_T(Z)G_T(Z)\right)$$

- T = rooted tree
  labelled by Γ
- $W_T(Z)$ = weight fct of graph T, proportional to
  $\prod_v DT(\alpha_v, Z)$
- $G_T(z,Z)$ is a graph integral defined recursively by

$$G_T(z,Z) := \int_{\gamma_T}\frac{dz'}{z'}\rho(z,z')X_0(z',Z)(e_{\gamma_T})\prod_{\{T'\}}G_{T'}(z',Z)$$

(jw Garcia, Fernandez, Filippini)

<u>Thm A</u> ($R\to 0$ scaling limit)

Take rescaling $z = Rt$ $Z \to RZ$
let $R \to 0$ the connections $\nabla(Z)$ converge to connections by BTL

$$\nabla^{BTL}(Z) = d - \left(\frac{Z}{t^2} + \frac{f(Z)}{t}\right)dt$$

(up to gauge) ↖ Joyce gen. fct.

<u>Thm B</u> ($R\to\infty$ scaling limit)

$Z \to RZ$ $R\to\infty$

The $G_T(z,Z)$ have a branch-cut behaviour as $Z\in \mathrm{Hom}(\Gamma,\mathbb{C})$ crosses critical locus

$$\text{JUMP OF}(G_T(z,RZ)) \underset{R\to\infty}{\sim} \sum_{T_\bullet}{}' G_{T_\bullet}(z,RZ) + \dots$$

<u>Thm C</u> ($R\to\infty$)

$$\sum_{w(T)=w}{}'\sum_{T_\bullet}{}' W(T_\bullet)\underbrace{\varepsilon(T_\bullet)}_{\pm 1} = N^{trop}(w)$$

where $w = w(T)$ trop. degree

Exercise session
Thursday
Mar 3rd

Thomas Reichelt
Vicky Hoskins

Gauss-Manin systems

$f: \mathbb{C}^x \to \mathbb{C}$
$x \mapsto x + \frac{1}{x}$

$$\mathcal{H}^0(f_+ \mathcal{O}_{\mathbb{C}^x}) = \frac{\Omega^1_{\mathbb{C}^x}[\partial_t]}{(d - df \wedge \partial_t)\Omega^0_{\mathbb{C}^x}[\partial_t]}$$

$$t(w \otimes \partial_t^n) = wf \otimes \partial_t^n - n w \otimes \partial_t^{n-1}$$

$$\partial_t(w \otimes \partial_t^n) = w \otimes \partial_t^{n+1}$$

$$\boxed{\text{Exc: Show that } \left((t^2-4)\partial_t + t\right) \cdot \left(\frac{dx}{x} \otimes \partial_t^0\right) = 0}$$

have in $\mathcal{H}^0(f_+ \mathcal{O}_{\mathbb{C}^x})$:

$$0 = (d - df \wedge \partial t)(x^k \otimes \partial_t^n) = k x^k \frac{dx}{x} \otimes \partial_t^n - x^k (x - \tfrac{1}{x}) \frac{dx}{x} \otimes \partial_t^{n+1}$$

$$\Rightarrow x \frac{dx}{x} \otimes \partial_t^0 = x(x - \tfrac{1}{x}) \otimes \partial_t \qquad k=1 \quad n=0$$

$k=-1 \quad n=0$

$$-\frac{1}{x} \frac{dx}{x} \otimes \partial_t^0 = \frac{1}{x}(x - \tfrac{1}{x}) \frac{dx}{x} \otimes \partial_t$$

$$t^2 \partial_t \left(\frac{dx}{x} \otimes \partial_t^0\right) = t^2 \left(\frac{dx}{x} \otimes \partial_t\right)$$

$$= t \cdot \left((x + \tfrac{1}{x}) \frac{dx}{x} \otimes \partial_t - \frac{dx}{x} \otimes \partial_t^0\right)$$

$$= (x + \tfrac{1}{x})^2 \frac{dx}{x} \otimes \partial_t - (x + \tfrac{1}{x}) \frac{dx}{x} \otimes \partial_t^0 - (x + \tfrac{1}{x}) \frac{dx}{x} \otimes \partial_t^0$$

$$-4\partial_t \left(\frac{dx}{x} \otimes \partial_t^0\right) = -4 \frac{dx}{x} \otimes \partial_t$$

$$t \cdot \left(\frac{dx}{x} \otimes \partial_t^0\right) = (x + \tfrac{1}{x}) \frac{dx}{x} \otimes \partial_t^0$$

$$(x - \tfrac{1}{x})^2 \frac{dx}{x} \otimes \partial_t - (x + \tfrac{1}{x}) \frac{dx}{x} \otimes \partial_t^0$$

$$x(x - \tfrac{1}{x}) \frac{dx}{x} \otimes \partial_t - \frac{1}{x}(x - \tfrac{1}{x}) \otimes \partial_t - (x + \tfrac{1}{x}) \frac{dx}{x} \otimes \partial_t^0$$

$$\Big\downarrow \text{I}$$

$$x \cdot \frac{dx}{x} \otimes \partial_t^0 + \frac{1}{x} \frac{dx}{x} \otimes \partial_t^0 - (x + \tfrac{1}{x}) \frac{dx}{x} \otimes \partial_t^0 = 0$$

1/2

Exc. 2: $\mathcal{H}^0(f_+ \mathcal{O}_{\mathbb{C}^x}) \xrightarrow{\partial_t} \mathcal{D}_{\mathbb{C}} / (t^2-4)\partial_t^2 + t\partial_t$

$\mathbin{/\!/}$

$\mathbb{C}[t] \oplus \mathcal{D}_{\mathbb{C}} / (t^2-4)\partial_t + t$

Show that $\mathcal{D}_{\mathbb{C}} / (t^2-4)\partial_t^2 + t\partial_t$ is ∂_t-localized

$$\mathbb{C}[t]\langle \partial_t, \partial_t^{-1} \rangle \otimes_{\mathbb{C}[t]\langle \partial_t \rangle} \mathcal{D}_{\mathbb{C}} / (t^2-4)\partial_t^2 + t\partial_t = \mathcal{D}_{\mathbb{C}} / (t^2-4)\partial_t^2 + t\partial_t$$

$\forall P \in \mathcal{D}_{\mathbb{C}} \quad \exists Q \in \mathcal{D}_{\mathbb{C}}$ s.t. $P = \partial_t \cdot Q$ [generally true for GKZ systems

Find Q s.t. $1 = \partial_t Q$ in $\mathcal{D}_{\mathbb{C}} / (t^2-4)\partial_t^2 - t\partial_t$

Claim: $Q = (4 - t^2)\partial_4 + t$

$$\partial_t \left((4-t^2)\partial_t + t\right) = (4 - t^2)\partial_t^2 - 2t\partial_t + t\partial_t + 1$$

$$= \underbrace{(4-t^2)\partial_t^2 - t\partial_t}_{=0} + 1 = 1$$

$\partial_t \cdot Q = 1$

$\partial_t Q \cdot t \neq t$ because have no right mult. in this module

however, it is enough to know Q_k s.t. $\partial_t Q_k = t^k$ as P in general looks like

$$P = t^k + \partial_t t^{k_1} + \partial_t^2 t^{k_2}$$

Hoskins

Exc: Let $Q = (V, A, h, t)$ be a quiver and fix $\theta \in \mathbb{Z}^V$, $\alpha \in \mathbb{Z}^V_{\geq 0}$

a) If W, W' are (θ, α)-semistable and $\mu_{\theta,\alpha}(W) < \mu_{\theta,\alpha}(W') = \frac{\theta(W')}{\alpha(W)}$ then $\operatorname{Hom}(W, W') = 0$

Pf: Suppose $f: W \to W'$ is non-zero

$$\begin{array}{c} W' \\ \cup\!\! \mid \\ 0 \neq \operatorname{Im} f \end{array}$$

$$\mu_{\theta,\alpha}(\operatorname{Im} f) \geq \mu_{\theta,\alpha}(W')$$

Observe $0 \neq \ker f \subseteq W \to \mu_{\theta,\alpha}(\ker f) \overset{(*)}{\geq} \mu_{\theta,\alpha}(W)$

$0 \to \ker f \to W \to \operatorname{Im} f \to 0$

$(*) \Rightarrow \mu_{\theta,\alpha}(W) \geq \mu_{\theta,\alpha}(\operatorname{Im} f) \geq \mu_{\theta,\alpha}(W')$ ↯

b) Show the existence & uniqueness of the HN filt.

$0 = W^{(0)} \subseteq W^{(1)} \subseteq \dots \subseteq W^{(r)} = W$

s.t. $W^j = W^{(j)}/W^{(j-1)}$ are (θ,α)-semistable

and $\mu_{\theta,\alpha}(W^1) < \dots < \mu_{\theta,\alpha}(W^r)$

Idea: construct iteratively: the first step $W^{(1)} \subseteq W$ is called the maximal destabilizing subrepr. we will show its existence and uniqueness

$$\mu_{\min,\theta,\alpha}(W) := \min_{0 \neq W' \subseteq W} \mu_{\theta,\alpha}(W')$$

Let $W^{(1)}$ be a subrep of W of max^l dim among those w minimal slope

Claim: $W^{(1)}$ is unique

Pf: If $W' \subseteq W$ and $\mu(W') = \mu_{\min}(W) \Rightarrow W' \subseteq W^{(1)}$

If not $W^{(1)} \subsetneq W^{(1)} + W' \subseteq W$

$$0 \to W' \cap W^{(1)} \hookrightarrow \underbrace{W' \oplus W^{(1)}}_{\substack{\mu(W' \oplus W^{(1)}) \\ \shortparallel \\ \mu_{\min}(W)}} \to W' + W^{(1)} \to 0$$

Must have $\mu_{\min}(W) = \mu(W' + W^{(1)})$ ↯

c) For Q acyclic show

$$Z_{\theta\alpha} : K(\operatorname{Rep} Q) \longrightarrow \mathbb{C}$$

$$W \longmapsto \sum_{v \in V}{}' (\theta_v + i\alpha_v) \dim W_v$$

is a Bridgeland stability condition with heart $\mathcal{A} = \operatorname{Rep} Q$ s.t.

$Z_{\theta,\alpha}$-semistab. $\Leftrightarrow$ $(\theta\alpha)$-semistab.

Claim: Q acyclic $\Rightarrow K(\operatorname{Rep} Q) \simeq \mathbb{Z}^V$

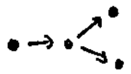
Pf: have gp homo

$$\underline{\dim} : K(\operatorname{Rep} Q) \to \mathbb{Z}^V$$

which is surjective as for each $v \in V$

$\exists$ simple rep $S_v = (\mathbb{C}^{\delta_{vw}}, w \in W, 0)$

As Q is acyclic, these are the only simple representations:



$\exists$ labelling

$V = \{1, \dots n\}$

s.t. if $\exists \alpha : V \to W$ then $V > W$

Then $\exists$ Jordan-Hölder filtration of any rep W of Q

$W^{(1)} \subseteq \dots \subseteq W^{(n)} = W$

s.t. $W^{(1)}/W^{(i-1)} \simeq S_i^{\oplus m_i}$ ▨

- For $W \neq 0$ we have $Z_{\theta,\alpha}(W) \in \mathbb{H}$ since $\sum \alpha_v \dim W_v > 0$

- To prove $Z_{\theta,\alpha}$-ss $\Leftrightarrow (\theta,\alpha)$-ss

$$\underset{\substack{\shortparallel \\ \phi'}}{\phi_{Z_{\theta,\alpha}}(W')} \leq \underset{\substack{\shortparallel \\ \phi}}{\phi_{Z_{\theta\alpha}}(W)} \Leftrightarrow \tan\left(\pi\phi + \frac{\pi}{2}\right)$$

$\uparrow$ tan increasing on $\frac{\pi}{2}, 3\frac{\pi}{2}$

$$\leq \tan\left(\pi\phi + \frac{\pi}{2}\right)$$

$\Updownarrow$ $\boxed{\tan(x + \frac{\pi}{2}) = {}^{(-1)}\cot(\alpha}$

$$\mu_{\theta,\alpha}(W') \geq \mu_{\theta,\alpha}(W)$$

Support property also works for

$$\|[W]\|^2 = \sum_{v \in V}{}' \dim W_v^2$$