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Quick Introduction to Stability Conditions

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Examples in dim 2 & 3

$\dim X = 1$ $\quad (Z, \mathrm{Coh}(X))$

$-\deg + \sqrt{-1}\,\mathrm{rk} \rightsquigarrow \mu(-) = \begin{cases} \dfrac{H^{n-1}\cdot c_1(-)}{H^n\cdot \mathrm{rk}(-)} & \text{if } \mathrm{rk} \neq 0 \\ +\infty & \text{otherwise} \end{cases}$

$\dim X = n$
H ample div.

H ample class on X $\quad \dim X = n$

$\alpha \in \mathbb{Q}_{>0} \rightsquigarrow \omega := \alpha H$

$B \in NS(X) \otimes \mathbb{Q}$

$$\mathrm{ch}^B(-) = e^{-B}\mathrm{ch}(-)$$

↳ Example: $\dim X = 2$ $\quad \mathrm{ch}^B(-) = \left(\mathrm{ch}_0(-), c_1(-) - B\,\mathrm{ch}_0(-), \mathrm{ch}_2(-) - B c_1(-) + \frac{B^2}{2}\mathrm{ch}_0(-)\right)$

$$\mu_{\omega,B}(-) := \begin{cases} \dfrac{\mathrm{ch}_1^B(-)\,\omega^{n-1}}{\omega^n\,\mathrm{ch}_0(-)} & \text{if } \mathrm{ch}_0(-) \neq 0 \\ +\infty & \text{otherwise} \end{cases}$$

$$\mathbb{T}_{\omega,B} = \{ E \in \mathrm{Coh}(X) : \mu_{\omega,B\,\min}(E) > 0 \}$$

$$\mathbb{F}_{\omega,B} = \{ E \in \mathrm{Coh}(X) : \mu_{\omega,B,\max}(E) \leq 0 \}$$

$$D^b(X)^{\leq 0} = \left\{ E \in D^b(X) : \begin{array}{l} \mathcal{H}^i(E) = 0 \quad i > 0 \\ \mathcal{H}^0(E) \in \mathbb{T}_{\omega,B} \end{array} \right\}$$

$$D^b(X)^{\geq 1} = \left\{ E \in D^b(X) : \begin{array}{l} \mathcal{H}^i(E) = 0 \quad i < 0 \\ \mathcal{H}^0(E) \in \mathbb{F}_{\omega,B} \end{array} \right\}$$

This defines a bounded T-structure (tilted t-structure)

↳ Happel-Reiten-Smalø

$\mathbb{A}_{\omega,B} = \{$ complexes with a non-trivial cohomologies in deg 0 $\in \mathbb{T}_{\omega,B}$ and deg -1 $\}$

$\|$

$D^b(X)^{\leq 0} \cap D^b(X)^{\geq 1}[1]$ $\quad (\in \mathbb{F}_{\omega,B})$

$$\Lambda_H := \mathrm{Im}(K(X) \longrightarrow \mathbb{Q}^{n+1}) \qquad n = \dim X$$

$$E \longmapsto (H^n \mathrm{ch}_0(E), H^{n-1}\mathrm{ch}_1(E), \dots, \mathrm{ch}_n(E))$$

$B = \beta H$ $\quad \beta \in \mathbb{Q}$

$$Z = -\int_X e^{-\sqrt{-1}\,\omega}\,\mathrm{ch}_B(-) = -\text{top degree of the cup product } \left(1, -i\omega, -\tfrac{\omega^2}{2}\right)(\text{—})$$

↳ $\dim X = 2$

$$Z(E) := -\left(\mathrm{ch}_2^B(E) - \frac{\omega^2}{2}\mathrm{ch}_0^B\right) + i\,\omega\,\mathrm{ch}_1^B(E).$$

claim

$$\sigma_{\omega,B} = (Z_{\omega,B}, \mathbb{A}_{\omega,B}) \in \mathrm{Stab}_{\Lambda_H}(D^b(X))$$

X sm proj surface$/\mathbb{C}$

↓ we want to prove that $Z(v(A)) \in \overline{\mathbb{H}}$ $\quad \forall\, 0 \neq A \in \mathbb{A}_{\omega,B}$

$A \in \mathbb{T}_{\omega,B}$ $\rightsquigarrow$ assume A $\mu_{\omega,B}$-semistable

1st case $\quad A$ torsion

2nd case $\quad A \quad \mu_{\omega,B}(A) > 0$ ✓

$$\mu_{\omega,B}(A) = \frac{\omega\cdot \mathrm{ch}_1^B(A)}{\omega^2\,\mathrm{ch}_0 A} > 0$$

$A \in \mathbb{F}_{\omega,B}[1]$

$\|$

$A'[1]$ where we can assume A' $\mu_{\omega,B}$-semistable

$\mu_{\omega,B}(A') \leq 0$

↳ $\mu_{\omega,B}(A') < 0$ $\quad \mathrm{Im}\,Z(A'[1]) > 0$

$\mu_{\omega,B}(A') = 0$

$$\mathrm{Re}\,Z(A'[1]) = \mathrm{ch}_2^B(A') - \underbrace{\frac{\omega^2}{2}\mathrm{ch}_0(A')}_{>0}$$

Bogomolov-Gieseker inequality

$$\operatorname{Re} Z(A'[1]) \le \frac{ch_1^B(A')^2}{2\, ch_0^B(A')} - \frac{\omega^2}{2} ch_0(A')$$

< 0 $\qquad \omega \cdot ch_1^B(A') = 0$

Hodge index theorem

$$\left.\begin{array}{l}\omega . E = 0 \\ \omega \text{ ample}\end{array}\right\} \Rightarrow E^2 < 0$$

$\dim X = 3$ Bayer-Macrì-Toda

$$Z_3(-) = -ch_3^B(-) + \frac{\omega^2}{2} ch_1^B(-) + i\left(\omega ch_2^B(-) - \frac{1}{6}\omega^3 ch_0(-)\right)$$

$$\underset{\mathcal{A}_{\omega,B}}{\underset{\cap}{\nu_{\omega,\beta}(A)}} = \begin{cases} \dfrac{\omega \cdot ch_2^B(A) - \frac{1}{6}\omega^3 ch_0(-)}{\omega^2 ch_1^B(A)} & \omega^2 ch_1^B(A) > 0 \\ +\infty & \text{otherwise}\end{cases}$$

$$\mathbb{T}'_{\omega,B} = \{ \mathcal{E} \in \mathcal{A}_{\omega,B} : \nu_{\omega,B\,min}(\mathcal{E}) > 0 \}$$

$$\mathbb{F}'_{\omega,B} = \{ \quad\text{------}\quad : \nu_{,B,max}(\mathcal{E}) \le 0 \}$$

$\mathcal{A}'_{\omega,B} \leftarrow$ 3 cohomologies

$= \langle \mathbb{F}'_{\omega,B}[1], \mathbb{T}'_{\omega,B} \rangle$

Q

$$(\underbrace{Z_{\omega,B}}_{:= Z_3}, \mathcal{A}'_{\omega,B}) \in \mathrm{Stab}_{\Lambda_H}(D^b(X)) \quad ?$$

Show: $\quad Z(\underset{\mathcal{A}'_{\omega,B}}{\underset{\cap}{\mathcal{E}(A)}}) \in \overline{\mathbb{H}}$ $\qquad$ Same arguments as before up to this case:

Assume: $\underset{A'[1]}{\overset{\|}{A}} \in \mathbb{F}'_{\omega,B}[1]$ $\nu_{\omega,B}(A) = 0$ $\quad A'$ - $\nu_{\omega,B}$-semistable

need to show $\quad -ch_3^B(A') + \frac{\omega^2}{2} ch_1^B(A') > 0$

Conj. (BMT) (BMS) $\quad A$ is $\nu_{\omega,B}$-semistable

$\nu_{\omega,B}(A) = 0$

$$\Rightarrow ch_3^B(A') \le \underset{\uparrow \text{ up to rescaling}}{\frac{\omega^2}{18}} ch_1^B(A')$$

Cases where the conjecture is true

$\mathbb{P}^3$ $\quad$ BMT + Macrì

quadrics, Schmidt

Abelian 3-fold (generic + ppav) $\quad$ Maciocia + Piyaratne

Any abelian 3fold + some compact proj 3folds

BMS $\quad$ (we prove more & expl. descr. conn. component

Li : Fano 3-folds of Pic # 1

Schmidt: Conjecture is false $\mathrm{Bl}_{pt}\mathbb{P}^3$ $\quad$ pullback of H is problematic

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Introduction to Non-Commutative Hodge structures III

Claude Sabbah

Fourier-Laplace transformation and nc Hodge str.

$(\mathcal{H}, \nabla)$ free $\mathcal{O}_{\mathbb{C}}$-mod w a merom. connection

$\mathcal{L} = \ker \nabla$ on $\mathbb{C}^*$ $S^1 = \{ |z| = 1 \}$

$\mathcal{L}_{\mathbb{R}} \rightsquigarrow$ Gluing procedure

bdl $\tilde{\mathcal{H}}$

$\hat{\mathcal{H}}$

Question: How to compute the Birkhoff-Grothendieck decomposition of $\tilde{\mathcal{H}}$ on $\mathbb{P}^1$

Start w var. of pol HS on punctured $\mathbb{A}^1 \setminus C$

[e.g. $f : Y \to \mathbb{A}^1$
smooth
quasi-proj.
tame

tame $:\Leftrightarrow$ $Rf_! \mathbb{C}_Y \xrightarrow{\alpha} Rf_* \mathbb{C}_Y$
+ isol. singularities
cone of α has constant cohomology

GM connection on $\mathbb{A}^1 \setminus C$

$C = f(\mathrm{Crit}(f))$

$gr^W_{n-1} H^{n-1}(f^{-1}(t))$]

Var of pol. HS of weight w

(V, ∇) flat hol. bdl on $\mathbb{A}^1 \setminus C$

- $\mathcal{V} = \ker \nabla$ $\mathcal{V}_{\mathbb{Q}}$: $Q_B : \mathcal{V}_{\mathbb{Q}} \otimes \mathcal{V}_{\mathbb{Q}} \to \mathbb{Q}_{\mathbb{A}^1 \setminus C}$ $(-1)^w$-symm.
- $F^p V \xrightarrow{\nabla} F^{p-1} V \otimes \Omega^1_{\mathbb{A}^1 \setminus C}$

whenever you restrict to a pt in $\mathbb{A}^1 \setminus C$
$\rightsquigarrow$ get a pol HS of weight w

Thm: Given such a variation $(F^{\cdot}V, \nabla, \mathcal{V}_{\mathbb{Q}}, Q_B, w)$
then its F.-L.-transform $((\mathcal{H}, \nabla), (\mathcal{L}_{\mathbb{Q}}, \mathcal{L}_{\mathbb{Q},\cdot}), \hat{F}(j_* Q_B), w+1)$
is a pure nc Hodge str. $\mathbb{Z}$ of weight $w+1$
pol

To Claude's knowledge this is the only known way to obtain nc HS

2) Betti side

$j : \mathbb{A}^1 \setminus C \hookrightarrow \mathbb{A}^1$

Fix $\theta_0 = \arg z_0$ to describe $\mathcal{L}_{\mathbb{Q},\theta_0}$, $\mathcal{L}_{\mathbb{Q},<c,\theta_0} \subset \mathcal{L}_{\mathbb{Q},\le c,\theta_0}$
$\theta_0 \in S^1$

$\overline{\mathbb{A}^1} := \mathbb{A}^1 \cup S^1_\infty$

Φ_{θ_0} family of closed subsets in $\mathbb{A}^1$

$S \in \Phi_{\theta_0} \overset{\mathrm{def}}{\Longleftrightarrow}$ ~~$\{\bar{S} \cap S^1_\infty \not\subset$~~
$\bar{S} \cap \{ \mathrm{Re}\, e^{i\theta}/z_0 \geq 0 \} = \phi$

$\mathcal{L}_{\mathbb{Q},\theta_0} = H^1_{\Phi_{\theta_0}}(\mathbb{A}^1, j_* \mathcal{V}_{\mathbb{Q}})$

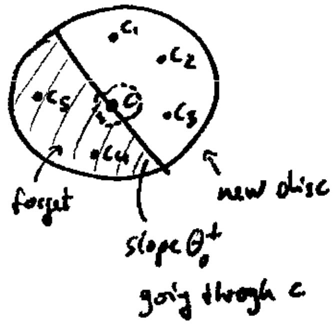
$= \mathbb{H}^1(\overline{\mathbb{A}^1}, \beta_! \alpha_* j_* \mathcal{V}_{\mathbb{Q}})$

where $\alpha : \mathbb{A}^1 \hookrightarrow \mathbb{A}^1 \cup (\text{open interval})$
$\beta : \mathbb{A}^1 \cup (\text{open interval}) \hookrightarrow \overline{\mathbb{A}^1}$

$\mathcal{L}_{\mathbb{Q},z_0} \otimes \mathcal{L}_{\mathbb{Q},-z_0} \longrightarrow \mathbb{Q}$
$\theta_0 + \pi$

$\Phi_{\theta_0} \cap \Phi_{\theta_0+\pi}$ = family of cpt sets in $\mathbb{A}^1$

$H^1_{\Phi_{\theta_0}} \otimes H^1_{\Phi_{\theta_0+\pi}} \xrightarrow{Q_B} H^2_c(\mathbb{A}^1, \mathbb{Q}) = \mathbb{Q}$



$\mathcal{L}_{\mathbb{Q},<c,\theta_0}$
$\cap$
$\mathcal{L}_{\mathbb{Q},\le c,\theta_0}$
$\cap$
$\mathcal{L}_{\mathbb{Q},\theta_0}$

de Rham side

jump iff $c = c_i$ for one i

adding / subtracting # vanishing cycles

$\mathrm{rk}\, \mathcal{L}_{\le c} / \mathcal{L}_{< c} = \dim \phi_{t-c}(j_* \mathcal{V}_{\mathbb{Q}})$

(V,∇) on $\mathbb{A}^1\setminus C$ $\rightsquigarrow$ get by intermediate ext. of (V,∇) by j.

Get a reg. hol $\mathcal{D}$-mod

Deligne ext. at $t=\infty$

M a $\mathbb{C}[t]\langle\partial_t\rangle$-mod which has reg. sing at C & ∞

$$\mathbb{C}[t]\langle\partial_t\rangle \longrightarrow \mathbb{C}[t]\langle\partial_t\rangle$$
$$\partial_t \rightsquigarrow t$$
$$t \rightsquigarrow -\partial_t$$

Laplace transformation

to be clear in notation use for RHS

$\mathbb{C}[\tau]\langle\partial_\tau\rangle$ — set $z=\tau^{-1}$

$\tau \mapsfrom \partial_t$, $\partial_\tau \mapsfrom -t$

$e^{-t\tau}$ kernel

$e^{-t/z}$

$$G = \mathbb{C}[\tau,\tau^{-1}]\langle\partial_\tau\rangle \otimes_{\mathbb{C}[\tau]\langle\partial_\tau\rangle} M$$

Lemma: G is a free $\mathbb{C}[\tau,\tau^{-1}]$-module of finite rank

$G \rightsquigarrow (\mathcal{G},\nabla)$ from yesterday for $\tau = z^{-1}$, $\tau^{-1}=z$

<u>Prop</u>: $(\mathcal{G},\nabla) \overset{\text{Del-Malgr}}{\Longleftrightarrow} (\mathcal{L}_{\mathbb{C}}, \mathcal{L}_{\mathbb{C}_i})$

$(V, F^\bullet V, \nabla)$ Schmid's theorem '73

$\Rightarrow F^p V$ extends as filtration of M by $\mathbb{C}[t]$-modules of finite type

$F^p M$

$\mathbb{C}[\partial_\tau]$-mod

$$\mathbb{C}[t]\langle\partial_t\rangle \;\text{—}\; M \xrightarrow{\widehat{loc}} G \;\leftarrow\; \mathbb{C}[\tau]\langle\partial_\tau\rangle$$

make $F^p M$ a bundle, i.e. a $\mathbb{C}[\tau]$-mod

Fix p_0 st. any $\underset{\ell\geq 0}{F^{p_0-\ell}} M = \sum_{i=0}^{\ell} \partial_t^i F^{p_0} M$

$$\mathcal{H} := \sum_{i=0}^{\infty} \underbrace{\partial_t^{-i}}_{(\tau^{-1})^i} \widehat{loc}(F^{p_0}M) \subseteq G$$

z^i

constant local system $\overset{FL}{\rightsquigarrow}$ dirac local syst. killed by $\tau \mapsto z\tau$

Monodromies on vanishing cycles

$$(\mathcal{G},\nabla) = \bigoplus \left(\mathcal{G}_{c_i}, \nabla_{c_i + \text{id}\frac{dz}{z^2}}\right)$$