Claus Hertling

# TERP-structures I

Cecotti-Vafa '91
Dubrovin '92

$tt^*$-equ

Simpson '88 $\boxed{\text{harm. bdls}}$
Sabbah '01
T. Mochizuki $\left.\begin{array}{l}\\ \\ \end{array}\right\}$ $\boxed{\text{polarized twistor D-modules}}$

$\boxed{\text{TERP str. H 02 Sevenheck 05}}$

$\boxed{\text{semi-infinite Hodge str. Barannikov 00}}$

$\boxed{\text{integrable twistor str. Sabbah 05}}$

$\boxed{\text{wild Hodge str. Sabbah 08}}$

$\boxed{\text{non-commutative Hodge str. Katzarkov-Kontsevich-Pantev 08}}$

**Def:** A polarized Hodge structure of weight $w \in \mathbb{Z}$

complex vector sp., $\dim H < \infty$

$H$
$\cup$
$H_{\mathbb{R}}$ $\quad$ $\mathbb{R}$-subspace $\quad H_{\mathbb{R}} \otimes \mathbb{C} = H$
$\cup$
$[H_{\mathbb{Q}}, H_{\mathbb{Z}}]$ $\quad$ $(-1)^w$-symmetric non-degen. pairing

$S: H_{\mathbb{R}} \times H_{\mathbb{R}} \to \mathbb{R}$

decreasing filtration on $H$, Hodge filtration

$\overline{F^{w-\bullet}}$ opposite to $F$ $\quad H = \bigoplus_{p \in \mathbb{Z}} F^p \cap \overline{F^{w-p}}$

eq: $\quad H = F^p \oplus \overline{F^{w+1-p}}$

$S(F^p, F^{w+1-p}) = 0$

$i^{2p-w} S(v, \bar{v}) > 0 \quad$ for $\quad v \in F^p \cap \overline{F^{w-p}} \quad v \neq 0$

$\rightsquigarrow$ Hermitian pairing

$h(a,b) = i^{2p-w} S(a, \bar{b})$

for $a \in F^p \cap \overline{F^{w-p}} \qquad b \in H$

**Def:** TERP str. of weight $w \in \mathbb{Z}$

$H \to \mathbb{C}$ hol vector bdl, $\mathcal{H} = \mathcal{O}(H)$ sheaf of hol sections

$\nabla$ flat hol connection on $H' := H|_{\mathbb{C}^*}$ with a pole of order $\leq 2$ at $0$

$H'_{\mathbb{R}} \to \mathbb{C}^*$ flat real subbundle of $H' \to \mathbb{C}^*$ with $H_{\mathbb{R},z} \otimes_{\mathbb{R}} \mathbb{C} = H'_z \quad \forall z \in \mathbb{C}^*$

$[H'_{\mathbb{Q}}: \text{TEQP}, \; H'_{\mathbb{Z}}, \text{TEZP}]$

$P$ a pairing $\quad P: H_z \times H_{-z} \to \mathbb{C} \quad \forall z \in \mathbb{C}^*$

flat, $\mathbb{C}$-bilinear, non-degenerate, $(-1)^w$-symmetric

$P(b,a) = (-1)^w P(a,b) \qquad a \in H_z, \; b \in H_{-z}$

$P: H'_{\mathbb{R},z} \times H'_{\mathbb{R},-z} \to i^w \cdot \mathbb{R}$ $\qquad$ [How to swap $a, b$?]

$[P: H'_{\mathbb{Z},z} \times H'_{\mathbb{Z},z} \to (2\pi i)^{-w} \mathbb{R}]$

$z^{-w} \cdot P: \mathcal{H}_0 \times \mathcal{H}_0 \to \mathcal{O}_{\mathbb{C},0}$ $\quad$ symm, non-degen.

$(H' \to \mathbb{C}^*, \nabla, H'_{\mathbb{R}}, P)$ $\quad$ top data

(extension $H \to \mathbb{C}$ of $H' \to \mathbb{C}^*$ to $0$, $\nabla$) $\quad$ transcendent datum $\rightsquigarrow F^\bullet$

$? \rightsquigarrow \overline{F^{w-\bullet}}$

$?? \rightsquigarrow F^\bullet$ and $\overline{F^{w-\bullet}}$ opposite

$??? \rightsquigarrow h$ pos. def.

$\boxed{\text{TERP = twistor extension, real pairing}}$

How to get TERP from usual Hodge str.?

$(H \; \mathbb{C}\text{-vs}, H_{\mathbb{R}} \; \mathbb{R}\text{-vs}, S)$ $\quad$ Rees Constr. $\dashrightarrow$ top data $\quad$ (TERP with log pole at 0)

$F^\bullet$

$F^\bullet$ and $\overline{F^{w-\bullet}}$ opp $\qquad$ pure TERP

$h$ pos. def. $\qquad$ pure pol TERP

**Next**

Explain $?, ??, ???$
Applications to sing. theory
Work of Cattani-Kaplan

Given a TERP structure, program:

① To construct an extension $\hat{H} \to \mathbb{P}^1$ of $H$ to $\infty$ $\quad \sim \overline{F^{w-\bullet}}$

② Def: The TERP is pure
$:\Leftrightarrow \hat{H} \to \mathbb{P}^1$ is a trivial bundle $\;|\sim$ pure HS

③ To construct a Hermitian form $h$ on $\Gamma(\mathbb{P}^1, \hat{H})$ (not nec. pos. def.)

Lemma: $h$ non-degen. $\Leftrightarrow$ the TERP is pure

④ Def. The TERP str. is pure & polarized
$\Leftrightarrow$ it is pure and $h$ pos. def.

①: $\gamma: \mathbb{P}^1 \to \mathbb{P}^1$
$z \mapsto 1/\bar z$

$\overline{\gamma^* H}$ hol [$\gamma$ is anti-hol, take anti-hol str. on fibres to have a hol. bdl]
vector bdl.
on $\mathbb{P}^1\setminus\{0\}$

For $z\in\mathbb{C}^*$ define $\tau: H_z \to \overline{H_{\gamma(z)}} = (\overline{\gamma^* H})_z$

$a \longmapsto \nabla$-flat shift of $\overline{z^{-w}\cdot a}$
along $z \to 1/\bar z$

Glue $H\to\mathbb{C}$ and
$\overline{\gamma^* H} \to \mathbb{P}^1\setminus\{0\}$ with $\tau$ to a hol. vector bdl $\hat H \to \mathbb{P}^1$

$(\hat H, \nabla)$ has a pole of order $\le 2$ at $\infty$

③ $\tau: \Gamma(\mathbb{P}^1, \hat H) \circlearrowleft$ $\tau^2 = id$ (new) real structure on $\Gamma(\mathbb{P}^1,\hat H)$

$\tau(\sigma)(z) := \tau(\sigma(1/\bar z))$

$z^{-w}P$ has constant values on $\Gamma(\mathbb{P}^1,\hat H)$

$S := z^{-w}P(\cdot,\tau\cdot): H_z \times H_{-1/\bar z} \to \mathbb{C}$

$h := (S$ on $\Gamma(\mathbb{P}^1,\hat H))$

Lemma: Give a pure TERP str.

$\rightsquigarrow \Gamma(\mathbb{P}^1,\hat H) \simeq \hat H_z \quad \forall z\in\mathbb{P}^1$

$\Rightarrow \exists$ two endom. $U$ and $Q$: $\Gamma(\mathbb{P}^1,\hat H)\circlearrowleft$

s.t. for $\sigma\in\Gamma(\mathbb{P}^1,\hat H)$ $\quad \nabla\sigma = (\frac{1}{z}U - Q - z\tau U\tau)(\sigma)\boxed{\frac{dz}{z}}$

$U \simeq [z^2\nabla_{\partial_z}]$ on $H_0$ $\quad \nabla_{z\partial_z}\sigma = (\dots)(\sigma)$

Q is called "new supersymmetric index"
of Cecotti-Fendley-Intriligator-Vafa '92

$Q^{*h} = Q$, $\quad Q = -\tau Q \tau$

h pos-def $\Rightarrow$ Q semisimple, eigenvalues in $\mathbb{R}$
symmetric around 0



Example: $f: Y\to\mathbb{C}$ $\quad$ Y affine alg. mfd
dim $Y = n$

f regular ~~hol~~ w isolated singularities
and in (some sense) tame at $\infty$.

$\mathbb{C} \supset \Sigma = \{u_1 \dots u_\ell\}$ = critval(f)

Flat cohom. bdl $\bigcup_{z\in\mathbb{C}\setminus\Sigma} H^{n-1}(f^{-1}(z),\mathbb{C})$ w.
Gauss-Manin connection

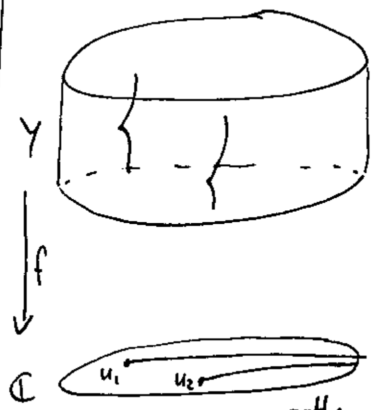
$\exists$ natural extension
$H^{gu} \to \mathbb{C}$ of this via hol. diff. form

Bad connection for generalization of HS:

Top data: the coho bdl on $\mathbb{C}\setminus\Sigma$ real subbdl

transcendent data $H^{gu}\to\mathbb{C}$ ← not constant under deform of f

→ oscillatory integrals over Lefschetz thimbles




paths
$\gamma_i: [0,\infty[\to\mathbb{C}$
$\gamma_i(s)$

Top. data: Fix $z\in\mathbb{C}^*$ (for a moment)

Fibre $H'_{\mathbb{Z},z} := Hom(\Lambda_{\mathbb{Z},z},\mathbb{Z})$
$\simeq \mathbb{Z}^\mu$

$\Lambda_{\mathbb{Z},z}$ := $\mathbb{Z}$-lattice generated by cohomology classes of Lefschetz thimbles in the direction z. $(+\infty)$

$:= H_n(Y, \{x\in Y \mid Re(\frac{f(x)}{z}) \gg 0\}, \mathbb{Z}) \simeq \mathbb{Z}^\mu$

Here, $\mu := \sum_{x\in crit(f)} \mu(f,x)$

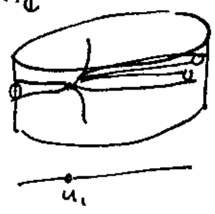
A Lefschetz thimble

$T_i(z) = \bigcup_{t\in Im(\gamma_i)} S_i(t)$ ← van. cycle

$H'_{\mathbb{Z}} := \bigcup_{z\in\mathbb{C}^*} H'_{\mathbb{Z},z}$
$\cap$
$H'_{\mathbb{R}}$
$\cap$
$H'_{\mathbb{C}}$

Pairing P: comes from the intersection form for Lefschetz thimbles

$I_{lef}: \Lambda_{\mathbb{Z},z} \times \Lambda_{\mathbb{Z},-z} \to \mathbb{Z}$
unimodular



Pham 84: $P := (-1)^{\frac{n(w-1)}{2}} \frac{1}{(2\pi i)^n} I_{\mathrm{Lef}}^* : H'_{\mathbb{Z},z} \times H'_{\mathbb{Z},z} \longrightarrow \frac{1}{(2\pi i)^n} \mathbb{Z}$

Transcendent datum:

extension of $H' \to \mathbb{C}^*$ to $0$ choose $\omega \in \Omega_Y^{n,\mathrm{alg}}$

Define glob. hol section $[\omega]$ in $H'$ by

$$\langle [\omega], [\Gamma(z)] \rangle := \int_{\Gamma(z)} e^{-f/z} . \omega$$

Such sections $\rightsquigarrow$ extension $H \to \mathbb{C}$ of $H' \to \mathbb{C}^*$

$\rightsquigarrow$ TEZP str and TERP str. of weight $w = n$

Claim: TEZP str. exist for $\hat{f}$ a defo of $f : Y \to \mathbb{C}$

<u>Thm</u> (Douai - Sabbah 03)
(Sabbah 05, CV 91) TERP$(f : Y \to \mathbb{C})$ is pure & polarized

# Introduction to Non-commutative Hodge structures I

Claude Sabbah

1) Germs of merom. connections of one variable $z$

(1) $\mathcal{H}$ : $\mathbb{C}\{z\}$ - free module w. $\nabla: \mathcal{H} \to \Omega^1(*0) \otimes_{\mathbb{C}\{z\}} \mathcal{H}$

(2) $(\mathcal{H}, \nabla)$ : free $\mathcal{O}_{\mathbb{C}}$-mod w. $\nabla: \mathcal{H} \to \Omega^1_{\mathbb{C}}(*0) \otimes_{\mathcal{O}_{\mathbb{C}}} \mathcal{H}$ — Cauchy thm

(3) $(\mathcal{H}, \nabla)$ : free $\mathcal{O}_{\mathbb{P}^1}(*\infty)$-module, $\nabla: \mathcal{H} \to \Omega^1_{\mathbb{P}^1}(*\{0,\infty\}) \otimes_{\mathcal{O}_{\mathbb{P}^1}(*\infty)} \mathcal{H}$ — Deligne's merom. extension at $\infty$

w. regular singularity at $\infty$

$\otimes_{\mathbb{C}[z]} \mathbb{C}\{z\}$

(4) $(\mathcal{H}, \nabla)$ : free $\mathcal{O}^{alg}_{\mathbb{P}^1}(*\infty)$-module, $\nabla$ --- alg. — variant of Serre's GAGA

(5) $(\mathcal{H}, \nabla)$ : free $\mathcal{O}^{alg}_{\mathbb{A}^1}$-mod, $\nabla$ --- — restriction to $\mathbb{A}^1$

(6) $(\mathcal{H}, \nabla)$ : free $\mathbb{C}[z]$-mod, $\nabla$... — $\Gamma(\mathbb{A}^1, \cdot)$

equiv. of categories

2) Numbers attached to $(\mathcal{H}, \nabla)$. Fix $\alpha \in \mathbb{R}$

Take $(\mathcal{H}, \nabla)$ as a $\mathcal{O}_{\mathbb{P}^1(*\infty)}$-mod w. connection

$\exists ! \ \mathcal{O}_{\mathbb{P}^1}$-free mod.

$V^\alpha \mathcal{H}$ s.t. $\nabla$ extends as a log connection on $V^\alpha \mathcal{H}$ and $\mathrm{Res}_\infty \nabla$ has eigenval w. real part in $[\alpha, \alpha+1[$

(Deligne's canonical extension w. $\alpha$-shift)

$V^\alpha \mathcal{H} \simeq \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_\mu)$ — Birkhoff-Grothendieck decomposition

$a_1 \geq \ldots \geq a_\mu$

**Def of $V^\alpha \mathcal{H}$: local at $\infty$**

$v$ local section of $\mathcal{H}$ near $\infty$ belongs to $V^0 \mathcal{H}$ if any sector centered at $\infty$ and any flat $\mathcal{H}, \nabla$ on this sector

The base change $v$ expressed in this flat basis with coeff having log growth at most in this sector

For $\alpha$, take $\left(\frac{1}{z}\right)^\alpha \left(\log \frac{1}{z}\right)^k$ — Deligne showed this defines a bundle

$\nu_\alpha := \# \{ i \mid a_i \geq 0 \}$

$\nu_\alpha = \nu_\alpha - \nu_{>\alpha}$

Set of pairs $(-\alpha, \nu_\alpha)$ w. $\nu_\alpha \neq 0$ is "Spectrum at $\infty$ of the connection $(\mathcal{H}, \nabla)$"

$f: Y \to \mathbb{A}^1$ — $Y$ smooth affine $\mathbb{C}$-variety, $f$ "tame" regular function

$z$ new variable

Consider $\mathcal{H} = \Omega^n(Y)[z] \big/ (zd + df \wedge \cdot)\,\Omega^{n-1}(Y)[z]$

$\mathbb{C}[z]$-free module $\quad \dim < \infty$ $\quad = \mu = \sum_{x \in \mathrm{Crit}(f)} \mathrm{Crit}_x(f) = \mu_x$

action of $\nabla_{\partial_z}$ induced by

$\partial_z - f/z^2 \circlearrowright \Omega^n(Y)[z]$

defines action on $\mathcal{H}$ as it descends to quotient

$$d + \frac{df}{z} = e^{-f/z} \circ d \circ e^{f/z}$$

gives relation to oscillatory integrals

3) Pole of order $\leq 2$

Assumption: non-ramified exponential type

$(\mathcal{H}, \nabla)$ $\quad \mathbb{C}\{z\}$-free mod w. $\nabla$-pole of order $\leq 2$

<u>Def</u> of nr exp type

$$\mathbb{C}[[z]] \otimes (\mathcal{H}, \nabla) \simeq \bigoplus_{c \in C \subset \mathbb{C}} \mathbb{C}[[z]] \otimes \left(\mathcal{H}_c, \nabla_c + c\,\mathrm{Id}\frac{dz}{z^2}\right)$$

$\nabla_c$ should have a regular singularity

$H^1$ on $\mathbb{C}^*$

$\mathbb{C}(\{z\}) \otimes_{\mathbb{C}\{z\}} \mathcal{H} = \mathcal{G}$ $\quad$ $\nabla$ extends naturally $\mathcal{G} \to \Omega^1 \otimes \mathcal{G}$

nr exp type $\Leftrightarrow$ $\mathbb{C}((z)) \otimes_{\mathbb{C}(\{z\})} (\mathcal{G}, \nabla) \simeq \bigoplus_{c \in C \subset \mathbb{C}} \mathbb{C}((z)) \otimes (\mathcal{G}_c, \nabla_c + c\,\mathrm{Id}$ ...

$(\mathcal{G}_c, \nabla_c)$ reg. sing.

4) Stokes filtration

$(\mathcal{G}, \nabla)$ of nr exp type

<u>Thm</u> (Deligne, Malgrange)

Category of $(\mathcal{G}, \nabla)$ of nr exp type is equivalent to the category of Stokes-filtered local systems $(\mathcal{L}, \mathcal{L}_\bullet)$ on $S^1(\arg z)$ of nr exp type

Can define notion of $\left|\frac{a}{z}\right|$-filtered local systems

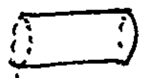
Stokes-filtered local systems attached to $(\mathcal{G},\nabla)$

$S^1$ $\quad \arg z = \theta$

on $S^1$ $\quad \mathcal{L}$ $\quad \operatorname{Ker}\nabla : H' \to H'$ $\quad\quad \mathbb{R}_+ := \mathbb{R}_{\geq 0}$

$\downarrow$

$\mathbb{C}^*$



$z=0$ $\quad z=1$ $\quad \xrightarrow{\text{polar coord.}}$

$S^1\times\mathbb{R}_+$

$\forall c\in\mathbb{C}$ have

$\mathcal{L}_{<0}\subset\mathcal{L}_{\leq c}\subset\mathcal{L}$

subsheaves

$\mathcal{A}^{mod}$ on $S^1\times\mathbb{R}_+ \overset{j}{\hookleftarrow} \mathbb{C}^*$

$\mathcal{A}^{mod}|_{\mathbb{C}^*} = \mathcal{O}_{\mathbb{C}^*}$

$\mathcal{A}^{mod}\subset j_*\mathcal{O}_{\mathbb{C}^*}$ of sections which have moderate growth $\quad$ e.g. $z^{\text{any power}}$

polynomial

$\mathcal{A}^{rd}\subset j_*\mathcal{O}_{\mathbb{C}^*}$ rapid decay $\quad$ e.g. $e^{1/z}$

only on open interval

$\mathcal{L}_{\leq c}$ : $\operatorname{Ker}\nabla_{\partial_z + c\frac{Id}{z^2}} : \mathcal{A}^{mod}\otimes\mathcal{G}\to\mathcal{A}^{mod}\otimes\mathcal{G}$

$\mathcal{L}_{<c}$ —— " —— $\mathcal{A}^{rd}$

$\mathcal{L}_{\leq c}/\mathcal{L}_{<c} = 0$ except for $c\in C$

and is a local system on $S^1$

<u>Stokes data</u>

Need a choice (of base point) : $\theta_0\in S^1$ generic wrt $C$

in the sense that $\forall c,c'\in C$, $\arg(c-c')\neq\theta_0\pm\pi/2$

C-graded vector spaces

$$\begin{tikzcd} \bigoplus_{c\in C} L_{1,c} \arrow[r, bend left, "S"] & \bigoplus_{c\in C} L_{2,c} \arrow[l, bend left, "S'"] \end{tikzcd}$$

maps $S$ upper triangular

$S'$ lower triangular

$c <_{\theta_0} c'$ if $\quad e^{(c-c')/z}$ has rapid decay near $\theta_0$

# A quick introduction to stability conditions

Paolo Stellari

## 1 - Definitions

Take $\mathcal{T}$ to be a triangulated category

$$\left(\mathcal{T} = D^b(X) = D^b \operatorname{Coh} X \quad \text{for } X \text{ sm. proj.} / \mathbb{C}\right)$$

Take $K(\mathcal{T})$ Grothendieck group of $\mathcal{T}$

Fix a finite rank lattice $\Lambda$ ($\cong \mathbb{Z}^n$) with a surjection

$$v : K(\mathcal{T}) \twoheadrightarrow \Lambda$$

$\|-\|$ a norm on $\Lambda \otimes \mathbb{R}$

### 1st definition of Bridgeland stability

**Def:** A Bridgeland stability condition on $\mathcal{T}$ is a pair $\sigma = (Z, \mathcal{P})$ where $Z : \Lambda \to \mathbb{C}$ linear, $\mathcal{P}$ a collection $\{\mathcal{P}(\phi), \phi \in \mathbb{R}\}$ where $\mathcal{P}(\phi)$ is a full subcategory of $\mathcal{T}$ s.t.

(1) $\mathcal{P}(\phi+1) = \mathcal{P}(\phi)[1]$

(2) $\phi_1 > \phi_2 \quad \operatorname{Hom}_{\mathcal{T}}(\mathcal{P}(\phi_1), \mathcal{P}(\phi_2)) = 0$

(3) If $0 \neq \mathcal{E} \in \mathcal{P}(\phi)$ then $Z(v(\mathcal{E})) \in \mathbb{R}_{>0}\, e^{i\pi\phi}$

(4) Harder-Narasimhan property
If $0 \neq \mathcal{E} \in \mathcal{T}$ then $\exists$ sequence of distinguished triangles

$$\begin{tikzcd}[column sep=small]
0=E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots & & E_n=\mathcal{E} \arrow[dl] \\
& A_1 \arrow[ul, dashed] & & A_2 \arrow[ul, dashed] & & & A_n \arrow[ul, dashed] &
\end{tikzcd}$$

with $A_i \in \mathcal{P}(\phi_i)$

$$\phi_1 > \phi_2 > \ldots > \phi_n.$$

(5) $\exists\, C > 0$ s.t. $\forall \phi \in \mathbb{R} \quad \forall\, 0 \neq \mathcal{E} \in \mathcal{P}(\phi)$

$$\|v(\mathcal{E})\| \leq C\, |Z(v(\mathcal{E}))|$$

support property (Kontsevich-Soibelman)

**Def:** An object $(0 \neq)\ \mathcal{E} \in \mathcal{P}(\phi)$ is called $\phi$-semistable

### t-structure on $\mathcal{T}$

**Def.** A t-structure on $\mathcal{T}$ is a pair of full additive subcategories $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 1})$ s.t.

(1) $\operatorname{Hom}_{\mathcal{T}}(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 1}) = 0$

(2) $\forall\, E \in \mathcal{T} \quad \exists$ distinguished triangle
$$E_1 \to E \to E_2 \to E_1[1]$$
where $E_1 \in \mathcal{T}^{\leq 0}$ and $E_2 \in \mathcal{T}^{\geq 1}$

(3) $\mathcal{T}^{\leq 0}[1] \subseteq \mathcal{T}^{\leq 0}$

Example $\quad \mathcal{T} = D^b(X)$

$$D^b(X)^{\leq 0} = \{\mathcal{E} \in D^b(X) : \mathcal{H}^i(\mathcal{E}) \cong 0 \ \ i > 0\}$$
$$D^b(X)^{\geq 0} = \{\mathcal{E} \in D^b(X) : \mathcal{H}^i(\mathcal{E}) \cong 0 \ \ i < 0\}$$

$$D^b(X)^{\leq 0} \cap D^b(X)^{\geq 1}[1] = \operatorname{Coh}(X)$$

**Def:** The heart of a t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 1})$ on $\mathcal{T}$ is the full (abelian!) subcategory $\mathcal{A} := \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 1}[1]$

Given $i \in \mathbb{Z}$

$$\mathcal{T}^{\leq -i} := \mathcal{T}^{\leq 0}[i]$$

$$\mathcal{T}^{\geq -i+1} := \mathcal{T}^{\geq 1}[i]$$

**Def.** A t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 1})$ on $\mathcal{T}$ is bounded if any $\mathcal{E} \in \mathcal{T}$ is contained in $\mathcal{T}^{\leq n} \cap \mathcal{T}^{\geq -n}$ for some $n \in \mathbb{Z}$

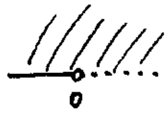
**Exercise**

If $\mathcal{A}$ is the heart of a bounded t-structure on $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 1})$ on $\mathcal{T}$ $\quad K(\mathcal{T}) = K(\mathcal{A})$

hint: filter object by cohomologies

**Def:** A stability condition on $\mathbb{T}$ is a pair $\sigma = (Z, \mathcal{A})$ where $\mathcal{A}$ = heart of $\mathbb{T}$-structure, $Z : \Lambda \to \mathbb{C}$ s.t.

(1) $Z(v(\mathcal{E})) \in \overline{\mathbb{H}} := \mathbb{R}_{>0}\, e^{(0,1]\pi i}$



$0 \neq \mathcal{E} \in \mathcal{A}$

(2) set $\lambda_\sigma(\mathcal{E}) := -\dfrac{\mathrm{Re}\, Z(v(\mathcal{E}))}{\mathrm{Im}\, Z(v(\mathcal{E}))}$

and define $\mathcal{E} \in \mathcal{A}$ to be $\lambda_\sigma$-semistable if $\forall F \hookrightarrow \mathcal{E}$ $\lambda_\sigma(F) \leq \lambda_\sigma(\mathcal{E})$

We require that for any $0 \neq \mathcal{E} \in \mathcal{A}$ $\exists$ a (HN)-filtration

$$0 = E_0 \hookrightarrow E_1 \hookrightarrow \dots \hookrightarrow E_{n-1} \hookrightarrow E_n = \mathcal{E}$$

s.t. $E_i/E_{i-1}$ is $\lambda_\sigma$-semistable

and

$$\lambda_\sigma(E_1/E_0) > \dots > \lambda_\sigma(E_n/E_{n-1})$$

(3) $\exists C > 0$ s.t. $\forall \mathcal{E} \in \mathcal{A}$ $\lambda_\sigma$-ss.

$$\| v(\mathcal{E}) \| \leq C\, | Z(v(\mathcal{E}))|$$

**Prop:** The two defin. are equivalent

$\hookrightarrow$ idea $\sigma = (Z, \mathcal{A})$ $\quad K(D^b) = K(\mathcal{A})$ so have $Z$

$P(\phi) = \{\lambda_\sigma$-semistable obj. in $\mathcal{A}$ of phase $\phi \in (0,1]\}$ if $\phi \in (0,1]$

If $\phi \in (n, n+1]$ $\quad P(\phi) := P(\phi - n)[n]$

Conversely, $\sigma = (Z, P)$ given

$\mathbb{T}^{\leq 0} := P(>0)$ = full subcat gen by $P(\phi)$, $\phi > 0$

$\mathbb{T}^{\geq 1} := P(\leq 0)$ = — " — $P(\phi)$, $\phi \leq 0$

**Properties** $\sigma = (Z, P)$ stab. cond.

(1) $P(\phi)$ is an abelian category

$\hookrightarrow$ $A \to B$ $\quad A, B \in P(\phi)$

The fact that 1st & 2nd def. are equiv.

$\Rightarrow$ $P(\phi) \subseteq$ heart of a bounded $t$-structure

**Def:** The objects in $P(\phi)$ without proper subobjects are $\sigma$-stable

(2) **Thm** (Bridgeland) $\mathrm{Stab}_\Lambda(\mathbb{T})$ = set of stab. conditions is a complex manifold, if not empty

**Q:** Is $\mathrm{Stab}_\Lambda(\mathbb{T}) \neq \emptyset$ ?

$\hookrightarrow$ $X = pt \Rightarrow \mathrm{Stab}(\mathbb{T}) \neq \emptyset$

$X = \mathrm{Spec}\, \mathbb{C}[\varepsilon]/\varepsilon^2 \quad \mathbb{T} = \mathrm{Perf}(X)$

$\mathrm{Stab}_\Lambda(\mathbb{T}) = \emptyset$ since no bounded $t$-structure

**Better Question:**

$X$ sm proj$/\mathbb{C}$ $\quad$ is $\mathrm{Stab}_\Lambda(D^b(X)) \neq \emptyset$?

(Specific choice $\Lambda = N(X) = K(X)/K(X)^\perp$ wrt $\chi$ Mukai pairing)

**Rmk:** We can rephrase the support property in the following equivalent (exc.!) way

$\exists$ a quadratic form $Q$ on $\Lambda \otimes \mathbb{R}$ s.t.

- the kernel of $Z$ is negative definite wrt $Q$
- For any $\sigma$-semistable object $\mathcal{E} \in \mathbb{T}$ we have $Q(v(\mathcal{E})) \geq 0$

**Rmk:** $\mathrm{Stab}_\Lambda(\mathbb{T})$ $(\neq \emptyset)$ carries two actions

- A left action of $\mathrm{Aut}(\mathbb{T})$

$$\mathrm{Aut}(\mathbb{T}) \times \mathrm{Stab}_\Lambda(\mathbb{T}) \to \mathrm{Stab}_\Lambda(\mathbb{T})$$

$$(F, \sigma = (Z, P)) \mapsto (Z', P')$$

where $Z'(v(\mathcal{E})) = Z(v(F^{-1}(\mathcal{E})))$

$P'(\phi) = F(P(\phi))$

- A right action of $\widetilde{Gl}^+(2,\mathbb{R})$ = univ. cover of $Gl^+(2,\mathbb{R})$

$\shortparallel$

$\{(M, \mathfrak{J}) : M \in Gl^+(2,\mathbb{R})\ \ \mathfrak{J} : \mathbb{R} \to \mathbb{R}$ s.t.

(1) $\mathfrak{J}$ is increasing with $\mathfrak{J}(\phi+1) = \mathfrak{J}(\phi) + 1$

(2) $M \cdot \exp(i\pi\phi) \in \mathbb{R}_{>0} \exp(i\pi \mathfrak{J}(\phi))$

$$\mathrm{Stab}_\Lambda(\mathbb{T}) \times \widetilde{Gl}^+(2,\mathbb{R}) \to \mathrm{Stab}_\Lambda(\mathbb{T})$$

$$(Z, P), (M, \mathfrak{J}) \longmapsto (Z', P')$$

$Z' = M^{-1} \circ Z \qquad P'(\phi) = P(\mathfrak{J}(\phi))$

<u>Example</u> $C$ smooth proj. curve $g(C) \geq 1$

$\Lambda = \mathbb{Z} \oplus \mathbb{Z}$ $\quad v : K(X) \to \mathbb{Z} \oplus \mathbb{Z}$

obvious map

$Z : \Lambda \to \mathbb{C}$

$(x, y) \mapsto -y + \sqrt{-1}\, x$ $\quad y = \deg$, $x = \mathrm{rk}$

$\sigma = (Z, \mathrm{Coh}(X)) \in \mathrm{Stab}_\Lambda(D^b(X))$

Actually $\mathrm{Stab}_\Lambda D^b(X) \simeq \widetilde{GL}_+(2, \mathbb{R}) \cdot \sigma$

If $\dim \geq 1 \Rightarrow$ slope stab does not give Bridgeland stability

Jacopo Stoppa

# Stability on tropical vertex groups

Rough idea:

Give a suitable category $\mathcal{C}$ and stab. cond. $\sigma$ on $\mathcal{C}$

$\Rightarrow$ get some canonical irregular connection on $\mathbb{P}^1$

topological part = central charge

transcendental part = it should be given by suitable DT invariants enumerating $\sigma$-stable objects in $\mathcal{C}$

$\rightarrow$ get isomonodromic families of connections by varying $\sigma$

$\rightarrow$ get interesting geometric objects out of flat sections

(Reineke, D-T, Joyce, Konts.-Soib., Bridgeland, Toledano-Laredo ..)

- Tropical vertex group $\mathbb{V}$ (Kontsevich-Soibelman, Gross-Siebert, Pandharipande)

Simplest version: start w. $\mathbb{C}^\times \times \mathbb{C}^\times$

Exc: $\mathrm{Aut}^{alg}(\mathbb{C}^\times \times \mathbb{C}^\times)$ fits in sequence

$$0 \to \mathbb{C}^\times \times \mathbb{C}^\times \to \mathrm{Aut}^{alg}(\mathbb{C}^\times \times \mathbb{C}^\times) \to GL(2,\mathbb{Z}) \to 1$$

rather boring, so look instead at

$$\mathrm{Aut}(\mathbb{C}^\times \times \mathbb{C}^\times \times \mathrm{Spec}\,\mathbb{C}[[t]]) \simeq \mathrm{Aut}_{\mathbb{C}[[t]]}(\mathbb{C}[x^{\pm1}, y^{\pm1}][[t]])$$

$$\mathbb{V} \subset \mathrm{Aut}\,\mathbb{C}[x^{\pm1}, y^{\pm1}][[t]]$$

Fix a function $f(x,y,t) = 1 + t x^a y^b g(x^a y^b, t)$

where $g \in \mathbb{C}[z][[t]]$ $\quad (a,b) \in \mathbb{Z}$

$\Rightarrow$ get $\theta_f \in \mathbb{V}$ s.t. $\theta_f(x) = x f^{-b}$

$\theta_f(y) = y f^a$

Eg $f_1 = (1+tx)^{\ell_1}$ $\quad \theta_{f_1}(x) = x$

$f_2 = (1+ty)^{\ell_2}$ $\quad \theta_{f_2}(y) = y\,(1+tx)^{\ell_1}$

$\ell_1, \ell_2 \geq 1$ $\quad \theta_{f_2}(x) = x\,(1+ty)^{-\ell_2}$

$\theta_{f_2}(y) = y$

$$\mathbb{V} = \overline{\{\text{subgrp gen. by } \theta_f\}}^{\,t\text{-adic topology}}$$

Exc: $(\theta_f)^{-1} = \theta_{f^{-1}}$ ~~formal~~

Exc: $\mathbb{V}$ is a group of formal symplectomorphisms of $\left(\mathbb{C}^\times \times \mathbb{C}^\times, \frac{dx}{x} \wedge \frac{dy}{y}\right)$

Commutators:

$$[\theta_f, \theta_g] := \theta_g^{-1} \theta_f \theta_g \theta_f^{-1}$$

today: just $[\theta_{f_1}, \theta_{f_2}]$

Ex: Prove that $\forall\, \ell_1\, \ell_2$ $\exists!$ factorization $[\theta_{f_1}, \theta_{f_2}] = \overrightarrow{\prod_{(a,b)}} \theta_{f_{a,b}}$ where $f_{(a,b)}$ is some function of $x^a y^b$

GPS: $\exists$ complete answer using some GW invariants

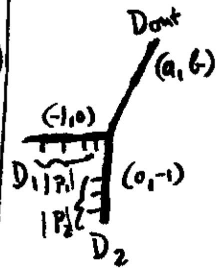
GW inv. of $\mathbb{P}^2(a,b,1)$ weighted proj. plane orbifold w. isol. orb. points

Combinatorial part

A pair of ordered partitions $P = (P_1, P_2)$

$X^o_{a,b} :=$ non-sing part of $\mathbb{P}(a,b,1)$

$X^o_{a,b}[P]$ = blowup of $X^o_{a,b}$ at points on divisors $D_1, D_2$




$$\beta_P = \pi^*\beta - \sum_{i,j}{}' p_{ij} E_{ij} \in H_2(X, \mathbb{Z})$$

$\beta$ gen. $H_2(X^o_{a,b}, \mathbb{Z})$

where we assume

$|P_1| = ka$ $\quad \beta.D_1 = ka$

$|P_2| = kb$ $\quad \beta.D_2 = kb$

$\beta.D_{out} = k$

Thm: $\exists$ well-def. GW invariant

$N_{(a,b)}[P] = \#^{vir}$ rational curves on $X^o_{a,b}[P]$ which are tangent to $D^o_{out}$ to order $k$ $\in \mathbb{Q}$

Thm (GPS): $\log(f_{a,b}) = \sum_{k\geq1} c_k^{a,b} (tx)^{ka} (ty)^{kb}$

where $c_k^{a,b} = \sum'_{|P_1|=ka} \sum'_{|P_2|=kb} k N_{a,b}[P]$

$\mathrm{len}(P_i) = \ell_i$

Idea of proof:

We will end up w a formula

$$C_k^{a,b} = k \sum_{|P_1| = ka}' \sum_{|P_2| = kb}' \sum_w' \frac{R_{p|w}}{Aut(w)} N^{trop}_{a,b}(w)$$

then $$N_{a,b}[P] = \sum_w' \frac{R_{p|w}}{Aut(w)} N^{trop}(w)$$

tropical invariants

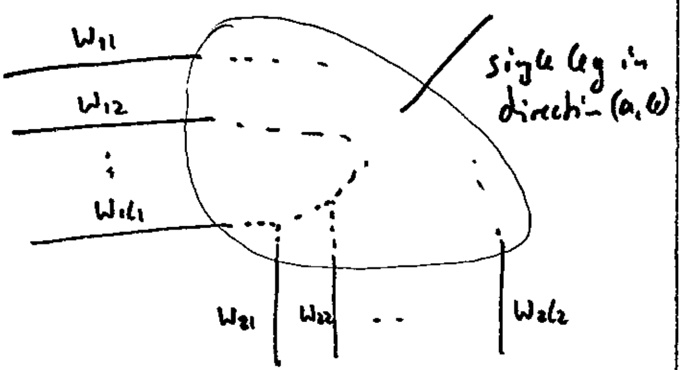
ramification factor

$w$ = weight vector = $(w_1, w_2)$

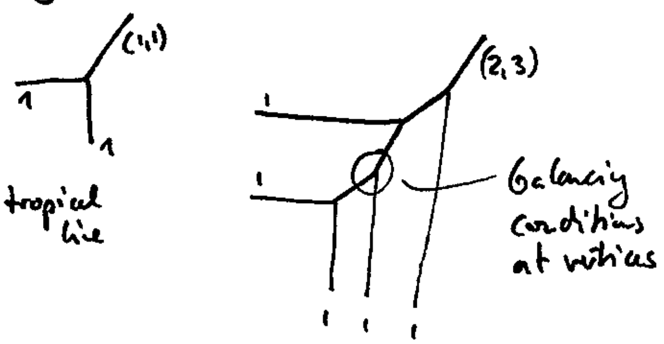
$0 \le w_{11} \le w_{12} \le \dots \le w_{1l_1}$

$0 \le w_{21} \le \dots \le w_{2l_2}$

$\rightsquigarrow N^{trop}(w)$ = count of rat'l trop curves in $\mathbb{R}^2$ w. boundary conditions given by $w$




Eg:



Feb 29

# Exercise session: Hodge theory

Hertling, Sabbah

$H$ $\mathbb{C}$-vs $\dim V = \mu < \infty$ $F^\cdot$ exh. decr. filtr.

$\widetilde{H} = V \times \mathbb{P}^1 \to \mathbb{P}^1$ is a flat hol vb on $\mathbb{P}^1$

connection $\nabla$, $\widetilde{\mathcal{H}} = \mathcal{O}(\widetilde{H})$

$F^p_{gl} := F^p \times \mathbb{P}^1 \to \mathbb{P}^1$ is flat subbdl of $\widetilde{H}$

$\mathcal{F}^p := \mathcal{O}(F^p_{gl})$

$H' := \widetilde{H}|_{\mathbb{C}^*} \to \mathbb{C}^*$

Define a new extension of $H' \to \mathbb{C}^*$ to a hol. vect. bdl. $H \to \mathbb{C}$ by its dual sheaf $\mathcal{H}$ of hol. sections

$$\mathcal{H} := \sum_{p \in \mathbb{Z}}{}' z^{-p} \mathcal{F}^p$$

$$\mathcal{H} \text{ is } z\text{-inv}^t \iff \mathcal{F}^{p+1} \subseteq \mathcal{F}^p$$

<u>Lemma</u> (a) $(H, \nabla)$ has a log pole at $0$

<u>Proof</u>: show $\nabla_{z\partial_z} : \mathcal{H}_0 \to \mathcal{H}_0$

$$\sigma \in z^{-p} \mathcal{F}^p \qquad \sigma = z^{-p} g(z)\, \sigma^{flat}$$

$$\nabla_{z\partial_z}(\sigma) = (-p)\sigma + z^{-p}(z\partial_z g(z))\, \sigma^{flat}$$

(b) The map $F^\cdot \mapsto (H \to \mathbb{C})$ gives 1-1 corresp between the two sets $\{F^\cdot$ an exh. decr. filtr. on $V\}$ and $\{$an extension $H \to \mathbb{C}$ of $H' \to \mathbb{C}^*$ w log pole at $0\}$

<u>Proof</u> Given $H \to \mathbb{C}$ w log pole at $0$

Any germ $\sigma \in \mathcal{H}_0$ takes the form $\sigma = \sum_{q \geq \alpha(\sigma)}{}' \underbrace{z^{q} v_q}_{\text{elem. sections}}$ for some $\alpha(\sigma) \in \mathbb{Z}$ where

$v_p \in V$

$v_{\alpha(\sigma)} \neq 0$

$$\boxed{(\nabla_{z\partial_z} + p)(z^{p} s_p) = 0}$$

Then $\alpha(\sigma)$ is the <u>order</u> of $\sigma$

$z^{-\alpha(\sigma)} v_{\alpha(\sigma)}$ is the principle part of $\sigma$

For $p \in \mathbb{Z}$ define

$F^p := \{ v \in V \mid \exists\, \sigma \in \mathcal{H}_0$ w $\alpha(\sigma) \leq p$ and $\sigma = \sum_{q \geq \alpha(\sigma)}' z^{q} v_q$ and $v_p = v\}$

$\overset{!}{=} \{ v \in V \mid z^{-p} v$ is the principle part of some section $\sigma \in \mathcal{H}_0\}$

$\overset{!}{=}$ With log pole & Vandermonde determinant

$$\sigma = \sum_{q \geq \alpha(\sigma)}{}' z^{-q} v_q$$

$$\nabla_{z\partial_z} \sigma = \sum{}' (-q) z^{q} v_q$$

$$(\nabla_{z\partial_z})^2 \sigma = \sum{}' (-q)^2 z^{-q} v_q$$

$$\vdots$$

$$(\nabla_{z\partial_z})^{p-\alpha(\sigma)} \sigma = \sum{}' (-q)^{p-\alpha(\sigma)} z^{-q} v_q$$

$$\det \begin{pmatrix} 1 & - & - & 1 \\ \alpha(\sigma) & \alpha(\sigma)+1 & \dots & p \\ \alpha(\sigma)^{p-\alpha(\sigma)} & \dots & & p^{p-\alpha(\sigma)} \end{pmatrix}$$

Compare $\sum_{p \in \mathbb{Z}}' z^{-p} \mathcal{F}^p|_{\mathbb{C}}$ and $\mathcal{H}$

$$\sum{}' z^{-p} \mathcal{F}^p|_{\mathbb{C}} \supset \mathcal{H}$$

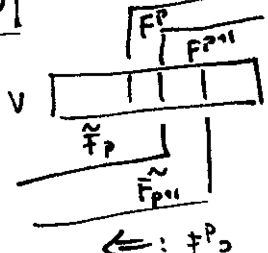
$$\sum_{p \in \mathbb{Z}}{}' \subseteq \mathcal{H} + z \sum_{p \in \mathbb{Z}}{}' z^{-p} \mathcal{F}^p$$

Lemma of Nakayama $\Rightarrow$ "$=$"

(c) Let $F^\circ$ be an exhaustive decreasing filtration and $\widetilde{F}_\bullet$ be an exhaustive incr. filtr.

They are opposite $\iff_{def}$ $\forall p \in \mathbb{Z}$ $V = F^p \oplus \widetilde{F}_{p-1}$

$$\underset{\text{Lemma}}{\iff} V = \bigoplus_p F^p \cap \widetilde{F}_p$$



$\Rightarrow$: $F^p = F^p \cap V = F^p \cap (F^{p+1} \oplus \widetilde{F}_p)$

$= F^{p+1} \oplus F^p \cap \widetilde{F}_p$

inductively $V = \bigoplus F^p \cap \widetilde{F}_p$

$\Leftarrow$: $F^p \underset{\text{obvious}}{\supset} \bigoplus F^p \cap \widetilde{F}_q$ and $\widetilde{F}_p \underset{\text{obv}}{\supset} \bigoplus_{q \leq p} F^q \cap \widetilde{F}_q$

Claim: $\overset{(1)}{=}$ and $\overset{(2)}{=}$

Claim $\Rightarrow V = F^p \oplus \tilde{F}_{p-1}$

Proof of $\underset{=}{(1)}$: suppose $v \in F^p \setminus \bigoplus_{q \geq p} F^q \cap \tilde{F}_q$

$v = \sum_q v_q$ w $v_q \in F^q \cap \tilde{F}_q$ and a $q < p$ w. $v_q \neq 0$ exists

$\Rightarrow v_q \in (F^q \cap \tilde{F}_q) \cap (F^p \cap \tilde{F}_p) = 0$ $\lightning$

d) Let $F^\bullet$ and $\tilde{F}_\bullet$ be as in c)

Def. hol vb $\hat{H} \to \mathbb{P}^1$ by $\hat{H}|_{\mathbb{C}} := H$ from $F^\bullet$

and $\hat{\mathcal{H}}|_{\mathbb{P}^1\setminus\{0\}} := \sum_{p\in\mathbb{Z}} z^{-p} \tilde{\mathcal{F}}_p|_{\mathbb{P}^1\setminus\{0\}}$ from $\tilde{\mathcal{F}}_\bullet$

It has log poles in $0$ & $\infty$

$F^\bullet$ and $\tilde{F}_\bullet$ are opposite $\Leftrightarrow$ $\hat{\mathcal{H}} = \mathcal{O}_{\mathbb{P}^1}^{\mu}$ ie

<u>Proof:</u> $\sigma \in \Gamma(\mathbb{P}^1, \hat{\mathcal{H}})$ $\quad \sigma = \sum_{\alpha(\sigma) \leq p \leq \beta(\sigma)} z^{-p} v_p$

$\Rightarrow$ all $z^{-p} v_p \in \Gamma(\mathbb{P}^1, \hat{\mathcal{H}})$

$v_p \in F^p \cap \tilde{F}_p$

$\Gamma(\mathbb{P}^1, \hat{\mathcal{H}}) = \sum_p z^{-p} F^p \cap \tilde{F}_p$ restricts at each $z \in \mathbb{P}^1$ to a basis of $\hat{H}_z$

$\Leftrightarrow$ $F^\bullet$ and $\tilde{F}_\bullet$ are opposite