Questions for the exercise sessions: Mainz Spring School 2016

For a quiver $Q = (V, A, h, t)$, let $\mathcal{R}\mathrm{ep}\, Q$ be the category of representations of Q over $\mathbb{C}$. For $\theta \in \mathbb{Z}^V$ and $\alpha \in \mathbb{N}^V_{>0}$, define the slope of a representation $0 \neq W \in \mathcal{R}\mathrm{ep}\, Q$ by

$$\mu_{\theta,\alpha}(W) := \frac{\sum_{v\in V} \theta_v \dim W_v}{\sum_{v\in V} \alpha_v \dim W_v}.$$

We say W is (θ, α)-*semistable* if for all subrepresentations $0 \neq W' \subset W$, we have

$$\mu_{\theta,\alpha}(W') \geq \mu_{\theta,\alpha}(W).$$

a) Prove that if W and W' are (θ, α)-semistable representations and $\mu_{\theta,\alpha}(W) < \mu_{\theta,\alpha}(W')$, then there are no non-zero morphisms from $W \to W'$.

b) Prove that every representation $0 \neq W \in \mathcal{R}\mathrm{ep}\, Q$ has a unique Harder–Narasimhan filtration with respect to (θ, α), which is a filtration

$$0 = W^{(0)} \subsetneq W^{(1)} \subsetneq W^{(2)} \subsetneq \cdots \subsetneq W^{(s)} = W$$

such that $W^i := W^{(i)}/W^{(i-1)}$ are (θ, α)-semistable and

$$\mu_{\theta,\alpha}(W^1) < \mu_{\theta,\alpha}(W^2) < \cdots < \mu_{\theta,\alpha}(W^s).$$

c) Now suppose that Q is acyclic (so that $K_0(\mathcal{R}\mathrm{ep}\, Q) \cong \mathbb{Z}^V$ is freely generated by the simple representations S_v with dimension vector $(\delta_{vw})_{v\in V}$). Prove that there is a group homomorphism $Z_{\theta,\alpha} : K_0(\mathcal{R}\mathrm{ep}\, Q) \to \mathbb{C}$ given by

$$Z_{\theta,\alpha}([W]) = \sum_{v\in V} (\theta_v + i\alpha_v) \dim W_v$$

where $i = \sqrt{-1}$. Furthermore, show that this determines a stability condition on $D^b(\mathcal{R}\mathrm{ep}\, Q)$ with heart $\mathcal{R}\mathrm{ep}\, Q$, such that the non-zero semistable objects in this heart are precisely the (θ, α)-semistable representations.

Hints

a) For a morphism $f : W \to W'$, use the fact that the kernel of f is a subrepresentation of W and the image is a subrepresentation of W'. Furthermore, for an exact sequence of representations $0 \to A \to B \to C \to 0$, check that

$$\mu_{\theta,\alpha}(A) \geq \mu_{\theta,\alpha}(B) \iff \mu_{\theta,\alpha}(B) \geq \mu_{\theta,\alpha}(C) \iff \mu_{\theta,\alpha}(A) \geq \mu_{\theta,\alpha}(C).$$

b) Show that W has a unique maximal (θ, α)-destabilising subrepresentation $W^{(1)} \subset W$ such that for all subrepresentations $0 \neq W' \subset W$, we have $\mu_{\theta,\alpha}(W^{(1)}) \leq \mu_{\theta,\alpha}(W')$

and if equality holds then $W' \subset W^{(1)}$. For the construction, let $W^{(1)}$ be a maximal dimensional subrepresentation of W with

$$\mu_{\theta,\alpha}(W^{(1)}) = \mu_{\min}(W) := \inf_{0 \neq W' \subset W} \mu_{\theta,\alpha}(W').$$

Check that if $W' \subset W$ and $\mu_{\theta,\alpha}(W') = \mu_{\min}(W)$, then $W' \subset W^{(1)}$. Then $W^{(1)}$ is (θ, α)-semistable and we can construct the filtration iteratively.

c) By [B] Proposition 5.3, to give a stability condition is equivalent to giving a bounded t-structure and a stability function on its heart with the Harder-Narasimhan property. Therefore, it suffices to show that $Z_{\theta,\alpha}$ is a group homomorphism such that for all $0 \neq W \in \mathcal{R}\text{ep}\, Q$, we have $Z_{\theta,\alpha}(W) \in H := \{r \exp(i\pi\theta) : r > 0, \theta \in (0,1]\}$ and every non-zero object has a unique Harder–Narasimhan filtration. The final statement follows from b), once one has shown that the semistability for $Z_{\theta,\alpha}$ coincides with (θ, α)-semistability (which is a basic exercise in trigonometry).

References

[B] T. Bridgeland, Stability conditions on triangulated categories.