Spring School on **Irregular Hodge structures and stability conditions**
Claus Hertling Mainz, 29.02.-04.03.2016

Two lectures on

TERP structures: definitions, examples, results.

Brief description of the contents:
Definition of TERP structures.
Examples from singularity theory.
Definitions of pure TERP structures and of pure and polarized TERP structures.
Results in the singularity case.
Nilpotent orbits of TERP structures and mixed TERP structures: a correspondence generalizing that for Hodge structures.
Stokes structures abstractly and in the case of singularities.
Comparison with the closely related noncommutative Hodge structures.
The base space of the unfolding of a singularity of type ADE is an atlas of Stokes structures.
Remarks on variations of TERP structures.

Two exercises

Exercise 1: Rees construction.
Data to start with: a complex vector space V, $\dim V =: \mu < \infty$,
and an exhaustive decreasing filtration $F^\bullet$ on V.
Construction: $\widetilde{H} := V \times \mathbb{P}^1 \to \mathbb{P}^1$ is a flat holomorphic vector bundle on $\mathbb{P}^1$ with flat holomorphic connection ∇, $\widetilde{\mathcal{H}} := \mathcal{O}(\widetilde{H})$ is the sheaf of holomorphic sections.
$F^p_{gl} := F^p \times \mathbb{P}^1 \to \mathbb{P}^1$ is a flat holomorphic subbundle of $\widetilde{H}$, $\mathcal{F}^p := \mathcal{O}(F^p_{gl})$.
Define the restriction $H' := \widetilde{H}_{|\mathbb{C}^*}$ to $\mathbb{C}^*$.
Define a new extension of $H' \to \mathbb{C}^*$ to a holomorphic vector bundle $H \to \mathbb{C}$ by its sheaf $\mathcal{H}$ of holomorphic sections:

$$\mathcal{H} := \sum_{p\in\mathbb{Z}} z^{-p}\mathcal{F}^p|_{\mathbb{C}}.$$

Exercise: Prove the following lemma.

Lemma: Let $(H' \to \mathbb{C}^*, \nabla)$ be as above.

(a) $(\mathcal{H}, \nabla)$ has a logarithmic pole at 0.

(b) The map $F^\bullet \mapsto (H \to \mathbb{C}, \nabla)$ gives a 1-1 correspondence between the two sets

$$\{F^\bullet \text{ an exhaustive decreasing filtration on } V\}$$
$$\text{and} \quad \{\text{an extension } H \to \mathbb{C} \text{ of } H' \to \mathbb{C}^* \text{ with logarithmic pole at } 0\}.$$

(c) Let $F^\bullet$ be an exhaustive decreasing filtration and $\widetilde{F}_\bullet$ be an exhaustive increasing filtration. They are *opposite* (to one another)

$$\iff_{\text{def.}} \quad \forall\, p \in \mathbb{Z} \quad V = F^p \oplus \widetilde{F}_{p-1}$$
$$\iff_{\text{lemma}} \quad \forall\, p \in \mathbb{Z} \quad V = \bigoplus_p F^p \cap \widetilde{F}_p.$$

(d) Let $F^\bullet$ and $\widetilde{F}_\bullet$ be as in (c). Define a holomorphic vector bundle $\widehat{H} \to \mathbb{P}^1$ by $\widehat{H}_{|\mathbb{C}} = H$ from $F^\bullet$ and

$$\widehat{\mathcal{H}}_{|\mathbb{P}^1-\{0\}} := \sum_{p\in\mathbb{Z}} z^{-p}\widetilde{F_p}|_{\mathbb{P}^1-\{0\}} \qquad \text{from } \widetilde{F}_\bullet.$$

It has logarithmic poles at 0 and ∞.

$F^\bullet$ and $\widetilde{F}_\bullet$ are opposite if and only if $\widehat{\mathcal{H}} \cong \mathcal{O}^\mu_{\mathbb{P}^1}$, i.e. $\widehat{H} \to \mathbb{P}^1$ is trivial as a holomorphic vector bundle.

Exercise 2: Two examples of variations of rank 2 regular singular TERP structures.

(a) Let $H' \to \mathbb{C}^* \times M$ with $M = \mathbb{P}^1$ (coordinates (z,t) on $\mathbb{C}\times\mathbb{C}$) be a flat vector bundle with basis A_1, A_2 of global flat sections.
Real structure $H'_\mathbb{R}$: $\overline{A_1} = A_2$, i.e. the basis $B_1 := A_1 + A_2, B_2 := i(A_1 - A_2)$ is real.
Pairing P:

$$P(\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, (A_1, A_2)) := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and thus} \quad P(\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, (B_1, B_2)) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Define $v_1 := z^{-1}A_1 + tA_2$ for $t \neq \infty$. Define an extension of $H' \to \mathbb{C}^* \times M$ to a bundle $H \to \mathbb{C} \times M$ as follows.
$\mathcal{H}_{|\mathbb{C}\times\mathbb{C}}$ is generated by v_1 and zA_2 (thus especially $A_1 \in \mathcal{H}_{|\mathbb{C}\times\mathbb{C}}$).
$\mathcal{H}_{|\mathbb{C}\times(\mathbb{P}^1-\{0\})}$ is generated by $t^{-1}z^{-1}A_1 + A_2 (= t^{-1}v_1$ for $t \neq \infty)$ and A_1 (thus especially $zA_2 \in \mathcal{H}_{|\mathbb{C}\times(\mathbb{P}^1-\{0\})}$).
This is a variation above the base space M of regular singular TERP structures of rank 2 and weight 0.
The map $\tau : H_z \to H_{1/\bar{z}}$ (complex conjugation and flat shift) takes on the sections the following form:

section	A_1	A_2	$z^{-1}A_1$	$v_1 = z^{-1}A_1 + tA_2$	zA_2
τ(section)	A_2	A_1	zA_2	$\tau(v_1) = zA_2 + \bar{t}A_1$	$z^{-1}A_1$

Exercise: For which $t \in M$ is its TERP structure $(H(t), \nabla, H'_\mathbb{R}(t), P)$ pure? Calculate $h := (P(.,\tau.)$ on $\Gamma(\mathbb{P}^1, \widehat{\mathcal{H}}(t)))$ for those t. For which t is it pure and polarized? Calculate $\widehat{\mathcal{H}}(t)$ for those t whose TERP structure is not pure. What is the signature of h in each region of pure TERP structures? Calculate $\mathcal{U}$ and $\mathcal{Q}$ there, i.e., calculate the matrices $\mathcal{U}^{mat}$ and $\mathcal{Q}^{mat}$ of $\mathcal{U}$ and $\mathcal{Q}$ with

$$\nabla_{z\partial_z}(\sigma_1, \sigma_2) = \left[\frac{1}{z}\mathcal{U} - \mathcal{Q} - z\tau\mathcal{U}\tau\right](\sigma_1, \sigma_2) = (\sigma_1, \sigma_2)\left[\frac{1}{z}\mathcal{U}^{mat} - \mathcal{Q}^{mat} - z(\tau\mathcal{U}\tau)^{mat}\right]$$

for a suitable basis (σ_1, σ_2) of $\Gamma(\mathbb{P}^1, \widehat{\mathcal{H}}(t))$.

(b) Data as in (a), except for the real structure, which is now defined by $\overline{A_1} = A_1, \overline{A_2} = A_2$, i.e. the basis A_1, A_2 is real.
Determine τ on the same sections as in (a) (and possibly some more). Find the regions of $t \in M$ whose TERP structures are pure, and calculate h and its signature there. Calculate $\mathcal{U}$ and $\mathcal{Q}$ there. Calculate $\widehat{\mathcal{H}}(t)$ for those t whose TERP structure is not pure.